Uniform formalism for description of dynamic, quantum and stochastic systems

Yuri A.Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1, Vernadskii Ave., Moscow, 119526, Russia.
e-mail: rylov@ipmnet.ru
Web site: $http://rsfq1.physics.sunysb.edu/\tilde{}rylov/yrylov.htm$
or mirror Web site:
$http://gasdyn-ipm.ipmnet.ru/\tilde{}rylov/yrylov.htm$

Abstract

The formalism of the particle dynamics in the space-time, where motion of free particles is primordially stochastic, is considered. The conventional dynamic formalism, obtained for the space-time, where the motion of free particles is primordially deterministic, seems to be unsuitable. The statistical ensemble of stochastic (or deterministic) systems is considered to be the main object of dynamics. At such an approach one can describe deterministic, stochastic and quantum particles by means of the uniform technique. The quantum particle is described as a stochastic particle, i.e. without a reference to the quantum principles. Besides, by means of this technique one can describe classical inviscid fluid. There are four different versions of the formalism: (1) description in Euler dynamic variables, (2) description in Lagrange dynamic variables, (3) description in terms of the generalized stream function, (4) description in terms of the wave function. The uniform formalism is purely dynamic. Even describing stochastic systems, it does not refer to probability and probabilistic structures. In relativistic case the uniform formalism can describe pair production and pair annihilation.

1 Introduction

The classical mechanics and the infinitesimal calculus had been created by Isaac Newton in the 17th century practically simultaneously. The ordinary differential equations were the principal tool of the classical mechanics. In the 18th and 19th centuries the development of the classical mechanics was carried out by means of modification of dynamic equations, when they were applied to new dynamic systems. All this time the conception of the event space (space-time) retained to be unchanged. In the Newtonian conception of the event space there are two independent invariants: space distance and time interval.

In the beginning of the 20th century Albert Einstein had discovered, that the dynamics may be developed not only by means of a modification of dynamic equations. The dynamics may be developed also by a modification of the event space. A. Einstein suggested and carried out the first two modifications of the event space. In the first modification, known as the special relativity, two Newtonian invariants (space and time) were replaced by one invariant – the space-time interval. After such a replacement one may speak about the space-time and the space-time geometry. The second modification of the event space was produced by A.Einstein ten years later. This modification is known as the general relativity. According to the general relativity the space-time geometry may be inhomogeneous, and this nonhomogeneity depends on the matter distribution in the space-time.

In the thirties of the 20th century it was discovered that the free particles of small mass move stochastically. *The motion of free particles depends only on the space-time properties.* It meant that for the explanation of the observed stochasticity one needs the next modification of the event space. The necessary third modification of the space-time geometry were to look rather exotic. As far as the stochasticity was different for the particles of different mass, the free motion of a particle must depend on the particle mass, i.e. the particle mass is to be geometrized. In the framework of the Riemannian geometry it was impossible. Besides, in the framework of the classical mechanics the particle motion is deterministic. If we want to explain the particle motion stochasticity by the space-time properties, we are to use such a space-time geometry, where the free particle motion be primordially stochastic. In the framework of the Riemannian geometry it was impossible. We did not know a geometry with such properties.

In the beginning of the 20th century we had the alternative: either space-time geometry with unusual exotic properties, or a refusal from the classical mechanics. We had no adequate space-time geometry. The alternative was resolved in favour of the quantum mechanics, which substituted the classical mechanics of the small mass particles. At such a substitution the principles of the classical mechanics were replaced by the quantum principles. Such a replacement of the classical principles by the quantum principles was a very complicated procedure, which was produced only for nonrelativistic phenomena.

The modification of the space-time geometry is more attractive from logical viewpoint, than the quantum mechanics, because it changes only space-time properties, but does not change classical principles of dynamics, whereas the quantum mechanics revises these principles. Unfortunately, the third modification of the space-time geometry was impossible in the first half of the 20th century.

The new conception of geometry, which made possible the third modification of the space-time [1], appeared only in the end of the 20th century . The new conception of geometry (known as T-geometry [2, 3, 4]) is very simple. It supposes that any space-time geometry is described completely by the world function [5], and then any space-time geometry can be obtained from the proper Euclidean geometry by means of a deformation (replacement of the Euclidean world function σ_{E} with the world function σ of the geometry in question in all definitions and relations of the proper Euclidean geometry). The T-geometry has unusual properties. In general, it is multivariant and nonaxiomatizable. It can be represented in coordinateless form. Discrete T-geometry and continuous one are described uniform.

In the new (nondegenerate) space-time geometry the particle mass is geometrized, the free particle motion is primordially stochastic, and the parallelism of vectors is absolute and intransitive, in general. Besides, parameters of the space-time depend on the quantum constant, and the statistical description of stochastic particles motion is equivalent to the quantum description. A use of the nondegenerate space-time geometry admits one to return to the classical mechanics of stochastic particles, eliminating quantum principles.

In general, the infinitesimal calculus, created for the Newtonian event space with the deterministic particle motion, disagrees with the space-time conception, where the free particle motion is primordially stochastic. One needs a new mathematical tool, which be in accordance with the nondegenerate space-time geometry. Construction of such a tool is a very difficult problem, and we shall not try to solve it. Instead, we *take from the new space-time geometry only the property of stochastic motion of free particles and try to describe it in the conventional Riemannian space-time*. In the Riemannian space-time the natural motion of free particles is deterministic. We try to formulate the conventional classical mechanics in such a form, where the stochastic particle motion be natural, whereas the deterministic particle motion be a special case of the stochastic motion, when the stochasticity vanishes.

Note that the conventional classical mechanics considers only deterministic particles, whose motion is described by dynamic equations (ordinary differential equations). One considers only some special cases of stochastic motion, referring to the probability theory in this consideration. The main object of the conventional classical mechanics is a single deterministic particle $\mathcal{S}_{\mathrm{d}}$. From viewpoint of the conventional classical mechanics the deterministic particle (dynamic system) and the stochastic particle (stochastic system) are conceptually different objects. The conceptual difference consists in the fact that there are dynamic equations for the dynamic system and there are no dynamic equations for the stochastic system. There is not even a collective concept with respect to concept of dynamic system and that of stochastic system.

If information on the deterministic particle $\mathcal{S}_{\mathrm{d}}$ is incomplete (for instance, if the initial conditions are known approximately), there are different versions of the particle $\mathcal{S}_{\mathrm{d}}$ motion. The particle motion is multiple-path. In this case we consider all these possible versions. One uses the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]$, which is the set of many independent particles $\mathcal{S}_{\mathrm{d}}$. Different elements $\mathcal{S}_{\mathrm{d}}$ of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]$ move differently, and motion of all these elements describe all possible motions of the particle $\mathcal{S}_{\mathrm{d}}$. The particle $\mathcal{S}_{\mathrm{d}}$ is a dynamic system, the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]$ is also a dynamic system. It means that there are dynamic equations for both the particle $\mathcal{S}_{\mathrm{d}}$ and the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]$. These dynamic equations describe the state evolution respectively of the particle $\mathcal{S}_{\rm d}$ and of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$.

Dynamic equations for $\mathcal{S}_{\rm d}$ and for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ are connected between themselves. For instance, if the dynamic system $\mathcal{S}_{\rm d}$ is a free nonrelativistic particle, the action $\mathcal{A}_{\mathcal{S}_{\rm d}}$ for $\mathcal{S}_{\rm d}$ has the form

$$\mathcal{A}_{\mathcal{S}_{\rm d}}\left[\mathbf{x}\right]=\int\frac{m}{2}\dot{\mathbf{x}}^{2}dt,\qquad\dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{1.1}$$

where $\mathbf{x}=\mathbf{x}\left(t\right)=\left\{x^{1}\left(t\right),x^{2}\left(t\right),x^{3}\left(t\right)\right\}$, and m is the particle mass.

The action for the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ of free independent particles $\mathcal{S}_{\rm d}$ is the sum of actions (1.1). It has the form

$$\mathcal{A}_{\mathcal{E}\left[\mathcal{S}_{\rm d}\right]}\left[\mathbf{x}\right]=\int\int\limits_{V_{\xi}}\frac{m}{2}\dot{\mathbf{x}}^{2}\rho_{0}\left(\boldsymbol{\xi}\right)dtd\boldsymbol{\xi},\qquad\dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{1.2}$$

where $\mathbf{x}=\mathbf{x}\left(t,\boldsymbol{\xi}\right)=\left\{x^{1}\left(t,\boldsymbol{\xi}\right),x^{2}\left(t,\boldsymbol{\xi}\right),x^{3}\left(t,\boldsymbol{\xi}\right)\right\}$, $\boldsymbol{\xi}=\left\{\xi_{1},\xi_{2},\xi_{3}\right\}$ are variables (Lagrangian coordinates), which label elements (particles) of the statistical ensemble. V_{ξ} is the region of variables $\boldsymbol{\xi}$. The quantity $\rho_{0}\left(\boldsymbol{\xi}\right)$ is the weight function. The quantity

$$N=\int_{V_{\xi}}\rho_{0}\left(\boldsymbol{\xi}\right)d\boldsymbol{\xi}\tag{1.3}$$

may be interpreted as the number of dynamic systems $\mathcal{S}_{\rm d}$, constituting the statistical ensemble.

Dynamic equations, generated by the actions (1.1) and (1.2) are similar

$$m\frac{d^{2}\mathbf{x}}{dt^{2}}=0,\qquad\mathbf{x}=\mathbf{x}\left(t\right)\tag{1.4}$$

$$\rho_{0}\left(\boldsymbol{\xi}\right)m\frac{d^{2}\mathbf{x}}{dt^{2}}=0,\qquad\mathbf{x}=\mathbf{x}\left(t,\boldsymbol{\xi}\right)\tag{1.5}$$

The dynamic systems $\mathcal{S}_{\rm d}$ and $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ are equivalent in the sense that one can obtain dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ from dynamic equations for $\mathcal{S}_{\rm d}$. Vice versa, one can obtain dynamic equations for $\mathcal{S}_{\rm d}$ from dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$.

What of dynamic systems $\mathcal{S}_{\rm d}$ and $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ is primary, and what is derivative? Conventionally, one supposes that the dynamic system $\mathcal{S}_{\rm d}$ is a primary fundamental object, whereas the statistical ensemble is considered usually as a secondary derivative object, because it is a more complicated object, consisting of $\mathcal{S}_{\rm d}$.

We suggest to consider the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ to be the primary object, whereas the single dynamic system $\mathcal{S}_{\rm d}$ is considered to be the secondary derivative object. Such an approach admits one to construct dynamics of stochastic systems.

Indeed, the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$ of independent stochastic systems $\mathcal{S}_{\rm st}$ is a dynamic system, although $\mathcal{S}_{\rm st}$ is a stochastic system. It means that there exist dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$, although there are no dynamic equations for $\mathcal{S}_{\rm st}$. Explanation of this surprising fact is as follows. When we construct the statistical ensemble of many stochastic systems, the regular features are accumulated, whereas random features are compensated. As a result, if the number of stochastic systems tends to infinity, we obtain the system, having only regular characteristics. In other words, we obtain a dynamic system.

Mathematically it looks as follows. We add some terms to the action (1.2). These terms are chosen in such a way, to describe the quantum stochasticity, generated by the properties of the space-time geometry. The supposed method of taking into account of this stochasticity leads to the quantum description, which has been well investigated. The action is written in the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}\left[\mathbf{x},\mathbf{u}\right]=\int\int_{V_\xi}\left\{\frac{m}{2}\dot{\mathbf{x}}^2+\frac{m}{2}\mathbf{u}^2-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}\rho_0\left(\boldsymbol{\xi}\right)dtd\boldsymbol{\xi},\qquad\dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{1.6}$$

The variable $\mathbf{x}=\mathbf{x}\left(t,\boldsymbol{\xi}\right)$ describes the regular component of the particle motion. The variable $\mathbf{u}=\mathbf{u}\left(t,\mathbf{x}\right)$ describes the mean value of the stochastic velocity component, $\hbar$ is the quantum constant. The second term in (1.6) describes the kinetic energy of the stochastic velocity component. The third term describes interaction between the stochastic component $\mathbf{u}\left(t,\mathbf{x}\right)$ and the regular component $\dot{\mathbf{x}}\left(t,\boldsymbol{\xi}\right)$. The operator

$$\boldsymbol{\nabla}=\left\{\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right\}\tag{1.7}$$

is defined in the space of coordinates $\mathbf{x}$. Dynamic equations for the dynamic system $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ are obtained as a result of variation of the action (1.6) with respect to dynamic variables $\mathbf{x}$ and $\mathbf{u}$.

To obtain the action functional for $\mathcal{S}_{\mathrm{st}}$ from the action (1.6) for $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$, we should omit integration over $\boldsymbol{\xi}$ in (1.6), as it follows from comparison of (1.2) and (1.1). We obtain

$$\mathcal{A}_{\mathcal{S}_{\mathrm{st}}}\left[\mathbf{x},\mathbf{u}\right]=\int\left\{\frac{m}{2}\dot{\mathbf{x}}^2+\frac{m}{2}\mathbf{u}^2-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}dt,\qquad\dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{1.8}$$

where $\mathbf{x}=\mathbf{x}\left(t\right)$ and $\mathbf{u}=\mathbf{u}\left(t,\mathbf{x}\right)$ are dependent dynamic variables. The action functional (1.8) is not well defined (for $\hbar\neq0$), because the operator $\boldsymbol{\nabla}$ is defined in some 3-dimensional vicinity of point $\mathbf{x}$, but not at the point $\mathbf{x}$ itself. As far as the action functional (1.8) is not well defined, one cannot obtain dynamic equations for $\mathcal{S}_{\mathrm{st}}$. By definition it means that the particle $\mathcal{S}_{\mathrm{st}}$ is stochastic. Setting $\hbar=0$ in (1.8), we transform the action (1.8) into the action (1.1), because in this case $\mathbf{u}=0$ in virtue of dynamic equations.

Dependence of the meaan stochastic velocity $\mathbf{u}$ on dependent variable $\mathbf{x}$, describing a regular component of the motion, and appearance of $\boldsymbol{\nabla}\mathbf{u}$ in the action functional for the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ are a formal sign of the particle stochasticity.

The quantum constant $\hbar$ has been introduced in the action (1.6), in order the description by means of the action (1.6) be equivalent to the quantum description by means of the Schrödinger equation [6]. If we substitute the term $-\hbar\boldsymbol{\nabla}\mathbf{u}/2$ by some function $f\left(\boldsymbol{\nabla}\mathbf{u}\right)$, we obtain statistical description of other stochastic system with other form of stochasticity, which does not coincide with the quantum stochasticity. In other words, the form of the last term in (1.6) describes the type of the stochasticity.

Although we cannot investigate the stochastic particle $\mathcal{S}_{\mathrm{st}}$, we can describe and investigate the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ of stochastic particles $\mathcal{S}_{\mathrm{st}}$, because $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ is well defined dynamic system (1.6). Investigation of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ admits one to investigate some average characteristics of the stochastic particle $\mathcal{S}_{\mathrm{st}}$. Information on $\mathcal{S}_{\mathrm{st}}$, obtained at investigation of $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$, is not a full information. One may obtain the mean velocity of the stochastic particle, the mean trajectories, the mean energy and some other average characteristics. However, one cannot obtain the velocity distribution and other more detailed characteristics of the stochastic particle. To obtain such detailed characteristics, one needs to use additional information on the stochastic particle properties (investigation of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ is insufficient for this goal). Nevertheless, the information, which is obtained from investigation of the statistical ensemble, appears to be valuable in many cases. For instance, one can show [6] that by a proper change of variables the action (1.6) is reduced to the action for the Schrödinger particle, i.e. to the action for the dynamic system, described by the Schrödinger equation.

Thus, there is a general approach to a description of stochastic particles, when the deterministic particle is considered to be a special case of stochastic particle (with vanishing stochasticity). To realize this approach, we are to *consider the statistical ensemble $\mathcal{E}\left[\mathcal{S}\right]$ as the primary object (basic object) of dynamics,* whereas the single system $\mathcal{S}$ is considered to be a derivative object of dynamics. Realizing this approach, it is useful to introduce a collective concept with respect to concept of dynamic system and that of stochastic system. We shall use the term "physical system". We shall speak about the statistical ensemble $\mathcal{E}\left[\mathcal{S}\right]$ of physical systems $\mathcal{S}$, and it is of no importance, whether $\mathcal{S}$ is the dynamic system, or the stochastic one. To stress that the dynamic system and the stochastic system are special cases of the physical system, we shall use the term "deterministic physical system" instead of the term "dynamic system" and the term "stochastic physical system" instead of the term "stochastic system". The fact that we can obtain dynamic equations for the dynamic system $\mathcal{S}$ and cannot obtain them for the stochastic system $\mathcal{S}$, will be considered as a special property of the statistical ensemble $\mathcal{E}\left[\mathcal{S}\right]$. There is a formal criterion, which admits one to determine, whether the physical systems $\mathcal{S}$, constituting the statistical ensemble $\mathcal{E}\left[\mathcal{S}\right]$, are stochastic systems. (Dynamic equations for statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]$ can be reduced to the system of ordinary differential equations, whereas for $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ such a reduction is impossible). The fact, that we cannot obtain a description (dynamic equations) for the single stochastic particle, is of no importance, because the basic object of dynamics is a statistical ensemble, and we can always obtain the description of the statistical ensemble.

Let us return to the action (1.6) and obtain dynamic equations for the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\text{st}}\right]$ of physical systems $\mathcal{S}_{\text{st}}$. Variation of (1.6) with respect to $\mathbf{u}$ gives

$$\begin{aligned}
\delta\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\text{st}}]}\left[\mathbf{x},\mathbf{u}\right] &= \int\int\limits_{V_\xi}\left\{m\mathbf{u}\delta\mathbf{u}-\frac{\hbar}{2}\boldsymbol{\nabla}\delta\mathbf{u}\right\}\rho_0\left(\boldsymbol{\xi}\right)dtd\boldsymbol{\xi} \\
&= \int\int\limits_{V_\mathbf{x}}\left\{m\mathbf{u}\delta\mathbf{u}-\frac{\hbar}{2}\boldsymbol{\nabla}\delta\mathbf{u}\right\}\rho_0\left(\boldsymbol{\xi}\right)\frac{\partial\left(\xi_1,\xi_2,\xi_3\right)}{\partial\left(x^1,x^2,x^3\right)}dtd\mathbf{x} \\
&= \int\int\limits_{V_\mathbf{x}}\delta\mathbf{u}\left\{m\mathbf{u}\rho+\frac{\hbar}{2}\boldsymbol{\nabla}\rho\right\}dtd\mathbf{x}-\int\oint\frac{\hbar}{2}\rho\delta\mathbf{u}dtd\mathbf{S}
\end{aligned}$$

where

$$\rho=\rho_0\left(\boldsymbol{\xi}\right)\frac{\partial\left(\xi_1,\xi_2,\xi_3\right)}{\partial\left(x^1,x^2,x^3\right)}=\rho_0\left(\boldsymbol{\xi}\right)\left(\frac{\partial\left(x^1,x^2,x^3\right)}{\partial\left(\xi_1,\xi_2,\xi_3\right)}\right)^{-1} \tag{1.9}$$

We obtain the following dynamic equation

$$m\rho\mathbf{u}+\frac{\hbar}{2}\boldsymbol{\nabla}\rho=0, \tag{1.10}$$

Variation of (1.6) with respect to $\mathbf{x}$ gives

$$m\frac{d^2\mathbf{x}}{dt^2}=\boldsymbol{\nabla}\left(\frac{m}{2}\mathbf{u}^2-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right) \tag{1.11}$$

Here d/dt means the substantial derivative with respect to time t

$$\frac{dF}{dt}\equiv\frac{\partial\left(F,\xi_1,\xi_2,\xi_3\right)}{\partial\left(t,\xi_1,\xi_2,\xi_3\right)}$$

Note that without a loss of generality we may set $\rho_0\left(\boldsymbol{\xi}\right)=1$, because by means of change of variables

$$\tilde{\xi}_1=\int\rho_0\left(\boldsymbol{\xi}\right)d\xi_1,\qquad\tilde{\xi}_2=\xi_2,\qquad\tilde{\xi}_3=\xi_3 \tag{1.12}$$

we obtain

$$\rho_0\left(\boldsymbol{\xi}\right)\frac{\partial\left(\xi_1,\xi_2,\xi_3\right)}{\partial\left(x^1,x^2,x^3\right)}=\frac{\partial\left(\int\rho_0\left(\boldsymbol{\xi}\right)d\xi_1,\xi_2,\xi_3\right)}{\partial\left(x^1,x^2,x^3\right)}=\frac{\partial\left(\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3\right)}{\partial\left(x^1,x^2,x^3\right)} \tag{1.13}$$

Resolving (1.10) with respect to $\mathbf{u}$, we obtain the equation

$$\mathbf{u}=-\frac{\hbar}{2m}\boldsymbol{\nabla}\ln\rho, \tag{1.14}$$

which reminds the expression for the mean velocity of the Brownian particle with the diffusion coefficient $D=\hbar/2m$.

Eliminating the velocity $\mathbf{u}$ from dynamic equations (1.11) and (1.14), we obtain the dynamic equations of the hydrodynamic type for the mean motion of the stochastic particle $\mathcal{S}_{\mathrm{st}}$

$$m\frac{d^2\mathbf{x}}{dt^2} = -\boldsymbol{\nabla} U_{\mathrm{B}}, \qquad U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right) = \frac{\hbar^2}{8m}\frac{\left(\boldsymbol{\nabla}\rho\right)^2}{\rho^2} - \frac{\hbar^2}{4m}\frac{\boldsymbol{\nabla}^2\rho}{\rho} \tag{1.15}$$

Here ρ is considered to be function of $t, \mathbf{x}$, and $\boldsymbol{\nabla}$ is the gradient in the space of coordinates $\mathbf{x}$.

If

$$\frac{\partial\left(\xi_1, \xi_2, \xi_3\right)}{\partial\left(x^1, x^2, x^3\right)} \neq 0 \tag{1.16}$$

and the relations $\mathbf{x} = \mathbf{x}\left(t, \boldsymbol{\xi}\right)$ can be resolved with respect to variables $\boldsymbol{\xi}$ in the form $\boldsymbol{\xi} = x\left(t, \mathbf{x}\right)$, dynamic equations (1.15) can be rewritten in the Euler form.

Using the relation (1.9), one can rewrite the designation

$$\mathbf{v} = \mathbf{v}\left(t, \mathbf{x}\right) = \frac{d\mathbf{x}}{dt} = \frac{\partial\left(\mathbf{x}, \xi_1, \xi_2, \xi_3\right)}{\partial\left(t, \xi_1, \xi_2, \xi_3\right)} = \frac{\partial\left(\mathbf{x}, \xi_1, \xi_2, \xi_3\right)}{\partial\left(t, x^1, x^2, x^3\right)}\frac{\partial\left(t, x^1, x^2, x^3\right)}{\partial\left(t, \xi_1, \xi_2, \xi_3\right)} \tag{1.17}$$

in the form

$$\rho\mathbf{v} = \rho\mathbf{v}\left(t, \mathbf{x}\right) = \rho\frac{d\mathbf{x}}{dt} = \rho_0\left(\xi\right)\frac{\partial\left(\mathbf{x}, \xi_1, \xi_2, \xi_3\right)}{\partial\left(t, x^1, x^2, x^3\right)} \tag{1.18}$$

Substituting (1.17) in (1.15), we obtain the Euler form of hydrodynamic equations

$$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial\mathbf{v}}{\partial t} + \left(\mathbf{v}\boldsymbol{\nabla}\right)\mathbf{v} = -\frac{1}{m}\boldsymbol{\nabla} U_{\mathrm{B}} \tag{1.19}$$

Instead of the definition (1.9) the quantity ρ in (1.15) and (1.19) is considered as $\rho = \rho\left(t, \mathbf{x}\right)$, which is determined from the continuity equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\left(\rho\mathbf{v}\right) = 0 \tag{1.20}$$

The continuity equation is fulfilled identically in force of relations (1.9) and (1.18).

Two equations (1.19), (1.20) describe evolution of the statistical ensemble of stochastic particles. Character of stochasticity is determined by the Bohm potential U_{B} [7], defined by the relation (1.15). Any reference to the stochastic velocity distribution or to some other probability distribution is absent. Influence of this distribution on the mean motion of the particles is described by the form of interaction (1.15). The situation reminds the case of the gas dynamics, where the action of the Maxwell velocity distribution on the gas motion is described by the internal gas energy. Of course, such a description is not comprehensive, however, it is sufficient for a description of the mean motion of the stochastic particle. As a result we obtain a *purely dynamic description* of the stochastic particle motion.

Equations for the ideal fluid may be described in terms of the wave function [6]. Irrotational flow of the fluid (1.15) is described by the Schrödinger equation for the free quantum particle. It means:

1. The statistical ensemble of free quantum nonrelativistic particles may be considered to be a statistical ensemble of stochastic particles, which is described by the action (1.6).

2. The wave function is simply a method of the ideal fluid description, but not a specific quantum object, defined by means of enigmatic quantum principles.

3. The quantum particles are stochastic particles, which may be described in terms of dynamics of physical systems, where the basic object is the statistical ensemble.

Thus, the dynamics of physical systems admits one to describe quantum effects without a reference to quantum principles, because the quantum particles, as well as stochastic ones are objects of classical dynamics of physical systems. Description of stochastic and quantum particles is the problem of the classical dynamics, where the *basic object of dynamics is the statistical ensemble.*

2 Dynamics of arbitrary physical systems

The action (1.6) for the statistical ensemble of free nonrelativistic stochastic particles may be easily generalized to the case of arbitrary stochastic systems. Let $\mathcal{S}_{\mathrm{d}}$ be a deterministic physical system having the finite number of the freedom degrees. The state of $\mathcal{S}_{\mathrm{d}}$ is described by the generalized coordinates $\mathbf{x}=\{x^1,x^2,...x^n\}$. The action has the form

$$\mathcal{A}_{\mathcal{S}_{\mathrm{d}}}[\mathbf{x}]=\int L_{\mathrm{d}}(t,\mathbf{x},\dot{\mathbf{x}},P)\,dt,\qquad \dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt} \tag{2.1}$$

where $\mathbf{x}=\mathbf{x}(t)$ and P are some parameters of the system (for instance, masses, charges, etc.)

Statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$ of dynamic systems $\mathcal{S}_{\mathrm{d}}$ is described by the action

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{d}}]}[\mathbf{x}]=\int\int_{V_{\boldsymbol{\xi}}} L_{\mathrm{d}}(t,\mathbf{x},\dot{\mathbf{x}},P)\,\rho_0(\boldsymbol{\xi})\,dtd^n\xi,\qquad \dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt} \tag{2.2}$$

where $\mathbf{x}=\mathbf{x}(t,\boldsymbol{\xi})=\{x^1(t,\boldsymbol{\xi}),x^2(t,\boldsymbol{\xi}),...x^n(t,\boldsymbol{\xi})\}$. The variables $\boldsymbol{\xi}=\{\xi_1,\xi_2,...\xi_n\}$ label elements $\mathcal{S}_{\mathrm{d}}$ of the statistical ensemble. The quantity $\rho_0(\boldsymbol{\xi})$ is the weight function. The number k of the labelling variables is chosen to be equal to the number n of generalized coordinates, in order one can to pass to the independent variables $t,\mathbf{x},$ resolving relations $\mathbf{x}=\mathbf{x}(t,\boldsymbol{\xi})$ in the form $\boldsymbol{\xi}=\boldsymbol{\xi}(t,\mathbf{x})$. If we are not going to pass to independent variables $t,\mathbf{x},$ the integer number $k>0$ may be chosen arbitrary.

If some disturbing agent influences on the deterministic system $\mathcal{S}_{\mathrm{d}}$, it turns into the stochastic system $\mathcal{S}_{\mathrm{st}}$ and the action (2.2) turns into the action $\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}$

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x},u]=\int\int_{V_{\boldsymbol{\xi}}} L(t,\mathbf{x},\dot{\mathbf{x}},P_{\mathrm{eff}}(u))\,\rho_0(\boldsymbol{\xi})\,dtd^n\xi,\qquad \dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt} \tag{2.3}$$

where $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$ and $u^k = \left\{u^k(t, \mathbf{x})\right\}$, $k = 0, 1, ...n$, are dependent variables. The new dependent variables u^k describe the mean value of the stochastic component of the generalized velocity $\dot{\mathbf{x}}$. It is supposed, that the disturbing agent changes the values of the parameters of dynamic system S_{d}. The Lagrangian $L(t, \mathbf{x}, \dot{\mathbf{x}}, P_{\mathrm{eff}}(u))$ for the statistical ensemble of the corresponding stochastic system $\mathcal{S}_{\mathrm{st}}$ is obtained from the Lagrangian $L_{\mathrm{d}}(t, \mathbf{x}, \dot{\mathbf{x}}, P)$ for the statistical ensemble of the dynamic system $\mathcal{S}_{\mathrm{d}}$ by means of the replacement [8]

$$P \rightarrow P_{\mathrm{eff}}(u) \tag{2.4}$$

in the expression (2.2). Passing to description of stochastic system $\mathcal{S}_{\mathrm{st}}$, we *do not introduce any probabilistic structures, and the descriptions remains to be purely dynamic.* Character of stochasticity is determined by the form of the change (2.4).

In the case, when the dynamic system $\mathcal{S}_{\mathrm{d}}$ is the free uncharged relativistic particle, the only parameter P is the particle mass m. If the stochastic agent is the distortion of the space-time geometry, the replacement (2.4) has the form

$$m \rightarrow m_{\mathrm{eff}} = \sqrt{m^2 + \frac{\hbar^2}{c^2}\left(g_{kl}\kappa^k\kappa^l + \partial_k\kappa^k\right)} \tag{2.5}$$

where c is the speed of the light, g_{kl} =diag$\{c^2, -1, -1, -1\}$ is the metric tensor,

$$\kappa^k = \frac{m}{\hbar}u^k, \qquad k = 0, 1, 2, 3 \tag{2.6}$$

and $u^k(t, \mathbf{x}) = u^k(x)$ is the mean value of the stochastic component of the particle 4-velocity. Here and later on there is a summation over repeating indices: $0 - 3$ for Latin indices and $1 - 3$ for Greek ones.

In the relativistic case the action for the statistical ensemble (2.3) has the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[x,\kappa] = -\int\int_{V_{\boldsymbol{\xi}}} mcK\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\rho_0(\boldsymbol{\xi})\,d\tau d\boldsymbol{\xi}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{d\tau} \tag{2.7}$$

$$K = \sqrt{1 + \lambda^2\left(g_{kl}\kappa^k\kappa^l + \partial_k\kappa^k\right)}, \qquad \lambda = \frac{\hbar}{mc} \tag{2.8}$$

where $x = \left\{x^k\right\} = \left\{x^k(\tau, \boldsymbol{\xi})\right\}$, $k = 0, 1, 2, 3$. The quantity g_{kl} =diag$\{c^2, -1, -1, -1\}$ is the metric tensor. The independent variables $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ label the particles of the statistical ensemble. The dependent variables $\kappa^k = \kappa^k(x)$, $k = 0, 1, 2, 3$ form some force field, connected with the stochastic component of the particle 4-velocity, and λ is the Compton wave length of the particle.

In the nonrelativistic approximation, one may neglect the temporal component $\kappa^0 = \frac{m}{\hbar}u^0$ with respect to the spatial one $\boldsymbol{\kappa} = \frac{m}{\hbar}\mathbf{u}$. Setting $\tau = t = x^0$ in (2.7), (2.8) we obtain instead of (2.7)

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x}, \mathbf{u}] = \int\int_{V_{\boldsymbol{\xi}}} \left\{-mc^2 + \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}\rho_0(\boldsymbol{\xi})\,dtd\boldsymbol{\xi}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \tag{2.9}$$

The action (2.9) coincides with the action (1.6) except for the first term, which does not contribute to dynamic equations.

In the relativistic case, varying (2.7) with respect to κ^i, we obtain the dynamic equations

$$\frac{\delta \mathcal{A}}{\delta \kappa^i} = -\lambda^2 \frac{mc\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\rho_0(\boldsymbol{\xi})}{K} g_{ik}\kappa^k + \lambda^2 \partial_i \frac{mc\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\rho_0(\boldsymbol{\xi})}{2K} = 0 \tag{2.10}$$

These equations are integrated in the form

$$\kappa = \frac{1}{2}\log \frac{C\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\rho_0(\boldsymbol{\xi})}{K}, \qquad C = \text{const} \tag{2.11}$$

where the quantity κ is a potential for the field κ^k

$$\partial_k \kappa = g_{kl}\kappa^l, \qquad k = 0, 1, 2, 3 \tag{2.12}$$

The dynamic equation (2.11) may be rewritten in the form

$$e^{2\kappa} = C \frac{\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\rho_0(\boldsymbol{\xi})}{\sqrt{1 + \lambda^2 g^{ls} e^{-\kappa}\partial_k \partial^k e^{\kappa}}}, \qquad C = \text{const} \tag{2.13}$$

which is an analog of nonrelativistic dynamic equation (1.14).

The principal difference between the nonrelativistic description (1.14) and the relativistic description (2.13) is as follows. The nonrelativistic equation (1.14) does not contain temporal derivatives, and the field $\mathbf{u}$ is determined uniquely by its source (the particle density ρ). The relativistic equation (2.13) contains temporal derivatives, and the κ-field $u^k = \hbar\kappa^k/m$ can exist without its source. The relativistic κ-field $u^k = \hbar\kappa^k/m$ can escape from its source. Besides, the κ-field changes the effective particle mass, as one can see from the relations (2.5) or (2.7), (2.8). If $\boldsymbol{\kappa}^2$ is large enough, or $\partial_k\kappa^k < 0$ and $\left|\partial_k\kappa^k\right|$ is large enough, the effective particle mass may be imaginary. In this case the mean world line may turn in the time direction, and this turn may appear to be connected with the pair production, or with the pair annihilation [8].

In the nonrelativistic case the mean stochastic velocity $\mathbf{u}$ may be eliminated and replaced by its source (the particle density ρ). In the relativistic case the κ-field has in addition its own degrees of freedom, which cannot be eliminated, replacing the κ-field by its source. The κ-field can travel from one space-time region to another.

The uniform formalism of dynamics (with the statistical ensemble as a basic object of dynamics) admits one to describe such a physical phenomena, which cannot be described in the framework of the conventional dynamic formalism, when the basic object is a dynamic system. In particular, one can describe the pair production effect, which cannot been described in the framework of the classical relativistic mechanics, as well as in the framework of the nonrelativistic quantum mechanics.

3 Statistical average physical system

Let $\mathcal{S}$ be a physical system and $\mathcal{E}[N,\mathcal{S}]$ be the statistical ensemble, consisting of N $(N\to\infty)$ physical systems $\mathcal{S}$. Let $\mathcal{A}_{\mathcal{E}[N,\mathcal{S}]}$ be the action functional for the statistical ensemble $\mathcal{E}[N,\mathcal{S}]$. As far as the statistical ensemble consists of identical independent systems, its action $\mathcal{A}_{\mathcal{E}[N,\mathcal{S}]}$ has the property

$$\mathcal{A}_{\mathcal{E}[aN,\mathcal{S}]}=a\mathcal{A}_{\mathcal{E}[N,\mathcal{S}]},\qquad a>0,\qquad a=\text{const},\qquad N,aN\gg 1 \tag{3.1}$$

Basing on this property, one may introduce such a statistical ensemble $\langle\mathcal{S}\rangle$, whose action $\mathcal{A}_{\langle\mathcal{S}\rangle}$ has the form

$$\mathcal{A}_{\langle\mathcal{S}\rangle}=\lim_{N\to\infty}\frac{1}{N}\mathcal{A}_{\mathcal{E}[N,\mathcal{S}]} \tag{3.2}$$

The deterministic physical system, whose action has the form (3.2), will be referred to as the *statistical average system* $\langle\mathcal{S}\rangle$. The physical system $\langle\mathcal{S}\rangle$ is a dynamic system, because it is deterministic and has the action $\mathcal{A}_{\langle\mathcal{S}\rangle}$. It is the average system, because its action $\mathcal{A}_{\langle\mathcal{S}\rangle}$ is the mean action for any system $\mathcal{S}$ of the statistical ensemble $\mathcal{E}[N,\mathcal{S}]$. According to definition (3.2) the system $\langle\mathcal{S}\rangle$ is the statistical ensemble $\mathcal{E}[N,\mathcal{S}]$, normalized to one system. In accordance with the property (3.1) the definition (3.2) of the action $\mathcal{A}_{\langle\mathcal{S}\rangle}$ is invariant with respect to transformation

$$N\to aN,\qquad a>0,\qquad a=\text{const} \tag{3.3}$$

Formally the statistical average system $\langle\mathcal{S}\rangle$ may be considered as a statistical ensemble consisting of one system $\mathcal{S}$. Nevertheless, according to (3.2) the statistical average system $\langle\mathcal{S}\rangle$ has statistical properties, because the action for $\langle\mathcal{S}\rangle$ is an action, constructed of the action for the statistical ensemble $\mathcal{E}[N,\mathcal{S}]$ with very large number N of elements $(N\to\infty)$.

Being a statistical ensemble, the statistical average system $\langle\mathcal{S}\rangle$ has some properties of the individual system $\mathcal{S}$. In particular, the energy E, the momentum $\mathbf{p}$ and other additive quantities of $\langle\mathcal{S}\rangle$ coincide respectively with the mean energy $\langle E\rangle$, the mean momentum $\langle\mathbf{p}\rangle$ and mean values of other additive quantities of the single system $\mathcal{S}$. In other words, in some aspects the statistical average system $\langle\mathcal{S}\rangle$ is perceived as a single system $\mathcal{S}$. On the other hand, the statistical average system $\langle\mathcal{S}\rangle$ does not coincide with $\mathcal{S}$, even if the single system $\mathcal{S}$ is a deterministic physical system.

Let, for instance, the single deterministic system $\mathcal{S}$ have n degrees of freedom. Let in the definition (3.2) the number N of elements of the statistical ensemble $\mathcal{E}[N,\mathcal{S}]$ be very large, but finite. In this case the statistical average system $\langle\mathcal{S}\rangle$ has nN degrees of freedom. The statistical average system $\langle\mathcal{S}\rangle$ may have alternative properties of the single system $\mathcal{S}$ simultaneously. For instance, let $\mathcal{S}$ be a single particle in the two-slit experiment. The individual particle $\mathcal{S}$ may pass only through one of two open slits, whereas the statistical average particle $\langle\mathcal{S}\rangle$ may pass through both slits simultaneously. The state of nk freedom degrees of $\langle\mathcal{S}\rangle$ correspond to the passage through one slit, whereas the state of $n(N-k)$ freedom degrees of $\langle\mathcal{S}\rangle$ correspond to the passage through another slit.

4 Methods of the statistical ensemble description

We shall consider four different methods of the statistical ensemble description: (1) description in Lagrangian coordinates, (2) description in Eulerian coordinates, (3) Hamilton-Jacobi description, (4) description in terms of the wave function. We demonstrate application of these methods in the example of nonrelativistic stochastic (quantum) particle, moving in the given external potential $V(\mathbf{x})$. In this case the action (1.6) takes the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x},\mathbf{u}] = \int\int_{V_\xi} \left\{ \frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x}) + \frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u} \right\} \rho_0(\boldsymbol{\xi})\, dt d\boldsymbol{\xi}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \quad (4.1)$$

where $\mathbf{x} = \mathbf{x}(t,\boldsymbol{\xi})$, $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$. After elimination of the variable $\mathbf{u}$ we obtain instead of (1.15)

$$m\frac{d^2\mathbf{x}}{dt^2} = -\boldsymbol{\nabla}V(\mathbf{x}) - \boldsymbol{\nabla}U_{\mathrm{B}} \quad (4.2)$$

where

$$U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right) = \frac{\hbar^2}{8m\rho}\left(\frac{(\boldsymbol{\nabla}\rho)^2}{\rho} - 2\boldsymbol{\nabla}^2\rho\right) = -\frac{\hbar^2}{2m\sqrt{\rho}}\boldsymbol{\nabla}^2\sqrt{\rho} \quad (4.3)$$

$$\rho = \rho_0(\boldsymbol{\xi})\left(\frac{\partial(x^1,x^2,x^3)}{\partial(\xi_1,\xi_2,\xi_3)}\right)^{-1} \quad (4.4)$$

To eliminate differentiation with respect to $\mathbf{x}$ and to write dynamic equations (4.2) in terms of the independent variables $t, \boldsymbol{\xi}$, we introduce the variable

$$R = \frac{\rho_0(\boldsymbol{\xi})}{\rho} = \frac{\partial(x^1,x^2,x^3)}{\partial(\xi_1,\xi_2,\xi_3)} = \det\left|\left|x^{\alpha,\beta}\right|\right|, \qquad \alpha,\beta = 1,2,3 \quad (4.5)$$

as a multilinear function of variables $x^{\alpha,\beta} \equiv \partial x^\alpha/\partial\xi_\beta$. We take into account that

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial\xi_\beta}{\partial x^\alpha}\frac{\partial}{\partial\xi_\beta} = \frac{1}{R}\frac{\partial R}{\partial x^{\alpha,\beta}}\frac{\partial}{\partial\xi_\beta}, \quad (4.6)$$

Then we obtain dynamic equations (4.2) in the form

$$m\ddot{x}^\alpha = -\frac{\partial V(\mathbf{x})}{\partial x^\alpha} + \frac{\hbar^2}{2mR}\frac{\partial R}{\partial x^{\alpha,\beta}}\frac{\partial}{\partial\xi_\beta}\left[\frac{1}{\sqrt{R}}\frac{\partial R}{\partial x^{\mu,\nu}}\frac{\partial}{\partial\xi_\nu}\left(\frac{1}{R}\frac{\partial R}{\partial x^{\mu,\sigma}}\frac{\partial}{\partial\xi_\sigma}\frac{1}{\sqrt{R}}\right)\right], \qquad \alpha = 1,2,3 \quad (4.7)$$

In terms of independent variables $t, \boldsymbol{\xi}$ the mean value $\mathbf{u}$ of the stochastic velocity has the form

$$u^\alpha(t,\boldsymbol{\xi}) = -\frac{\hbar}{2m\rho_0(\boldsymbol{\xi})}\frac{\partial R}{\partial x^{\alpha,\beta}}\frac{\partial}{\partial\xi_\beta}\frac{\rho_0(\boldsymbol{\xi})}{R} \quad (4.8)$$

Thus, in the Lagrangian variables $t, \boldsymbol{\xi}$ the dynamic equations for the statistical ensemble of stochastic (quantum) particles are rather bulky. However, if the particles are deterministic, and $\hbar = 0$, dynamic equations (4.7) turn to the ordinary differential equations

$$m\ddot{\mathbf{x}} = -\boldsymbol{\nabla} V(\mathbf{x}), \qquad \mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi}) \tag{4.9}$$

If in the dynamic equations (4.9) the variable $\mathbf{x}$ does not depend on $\boldsymbol{\xi}$, they are dynamic equations for the single classical particle. In order to pass from the equation (4.9) for the single particle, described by $\mathbf{x} = \mathbf{x}(t)$, to the dynamic equations (4.7), i.e. "to quantize the classical particle", one needs to consider statistical ensemble (replace $\mathbf{x} = \mathbf{x}(t)$ by $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$) and to add two last terms, containing the quantum constant. Thus, the conventional quantization may be considered as some dynamic procedure, introducing additional terms in the action of the statistical ensemble. One needs no quantum principles for such a quantization, because the concept of the wave function does not used here, (the quantum principles are needed only for explanation, what is the wave function)

If the relation (1.16) takes place, and relations $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$ can be resolved with respect to variables $\boldsymbol{\xi}$ in the form $\boldsymbol{\xi} = x(t, \mathbf{x})$, dynamic equations (4.9) can be rewritten in the Eulerian variables in the form (1.19), (1.20)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{m}\boldsymbol{\nabla}\left(U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right) + V(\mathbf{x})\right) \tag{4.10}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho\mathbf{v}) = 0 \tag{4.11}$$

where $\rho = \rho(t, \mathbf{x})$, $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$. In the Eulerian coordinates the dynamic equations for the statistical ensemble are simpler, than those in the Lagrangian coordinates. At the same time they are rather demonstrable. To obtain the mean trajectories of stochastic particles, one needs to solve at first the dynamic equations (4.10), (4.11). When the variables $\rho = \rho(t, \mathbf{x})$, $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ become known, one needs to solve ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(t, \mathbf{x}) \tag{4.12}$$

To obtain dynamic equations in terms of the generalized stream function (the Hamilton-Jacobi form), we are to integrate dynamic equations (4.10), (4.11) and formulate dynamic equations in terms of hydrodynamic potentials $\boldsymbol{\xi}$ (Clebsch potentials [10, 11]). The hydrodynamic potentials $\boldsymbol{\xi}$ may be considered as the generalized stream function, because they have the property of the stream function: (1) they label the world lines of the fluid particles and (2) some combination of the derivatives of $\boldsymbol{\xi}$ satisfy the continuity equation identically at any values of $\boldsymbol{\xi}$. (See for details [12]).

To produce integration of dynamic equations, we return to the action (4.1), which has now the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x}] = \int\int_{V_{\boldsymbol{\xi}}} \left\{\frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 - V(\mathbf{x}) - U_{\mathrm{B}}\right\}\rho_0(\boldsymbol{\xi})\, dt d\boldsymbol{\xi} \tag{4.13}$$

where $\mathbf{x} \equiv \mathbf{x}(t, \boldsymbol{\xi})$. The variables ρ and $U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right)$ are defined by the relation (4.3), (4.4).

To transform the action (4.13) to independent variables $x = \left\{x^k\right\} = \{t, \mathbf{x}\}$, we use the parametric representation of the mean world lines $\mathbf{x} \equiv \mathbf{x}(t, \boldsymbol{\xi})$. Let

$$x^k = x^k\left(\xi_0, \boldsymbol{\xi}\right) = x^k(\xi), \qquad k = 0, 1, 2, 3 \tag{4.14}$$

where $\xi = \{\xi_k\} = \{\xi_0, \boldsymbol{\xi}\}$, $k = 0, 1, 2, 3$. The shape of the world line is described by x^k, considered as a function of ξ_0 at fixed $\boldsymbol{\xi}$. The action (4.13) can be rewritten in the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[x] = \int\limits_{V_{\boldsymbol{\xi}}} \left\{\frac{m}{2}\left(\frac{\partial \mathbf{x}}{\partial \xi_0}\right)^2 \left(\frac{\partial x^0}{\partial \xi_0}\right)^{-1} - \left(V(x) + U_{\mathrm{B}}\right)\frac{\partial x^0}{\partial \xi_0}\right\}\rho_0(\boldsymbol{\xi})\, d^4\xi, \tag{4.15}$$

Let us consider the variables $\xi = \{\xi_k\}$, $k = 0, 1, 2, 3$ as dependent variables and variables $x = \left\{x^k\right\}$ as independent ones. We consider the Jacobian

$$J = \frac{\partial\left(\xi_0, \xi_1, \xi_2, \xi_3\right)}{\partial\left(x^0, x^1, x^2, x^3\right)} = \det\left|\left|\xi_{l,k}\right|\right|, \qquad \xi_{l,k} \equiv \frac{\partial \xi_l}{\partial x^k} \qquad l, k = 0, 1, 2, 3 \tag{4.16}$$

as a four-linear function of variables $\xi_{l,k} \equiv \partial_k \xi_l$, $l, k = 0, 1, 2, 3$. We take into account that

$$\frac{\partial x^k}{\partial \xi_0} = \frac{\partial\left(x^k, \xi_1, \xi_2, \xi_3\right)}{\partial\left(\xi_0, \xi_1, \xi_2, \xi_3\right)} = \frac{\partial\left(x^k, \xi_1, \xi_2, \xi_3\right)}{\partial\left(x^0, x^1, x^2, x^3\right)}\frac{\partial\left(x^0, x^1, x^2, x^3\right)}{\partial\left(\xi_0, \xi_1, \xi_2, \xi_3\right)} = J^{-1}\frac{\partial J}{\partial \xi_{0,k}} \tag{4.17}$$

and

$$\rho = \rho_0(\boldsymbol{\xi})\frac{\partial\left(\xi_1, \xi_2, \xi_3\right)}{\partial\left(x^1, x^2, x^3\right)} = \rho_0(\boldsymbol{\xi})\frac{\partial\left(x^0, \xi_1, \xi_2, \xi_3\right)}{\partial\left(x^0, x^1, x^2, x^3\right)} = \rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial \xi_{0,0}} \tag{4.18}$$

The action (4.15) takes the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\xi] = \int\limits_{V_x} \left\{\frac{m}{2}\left(\frac{\partial J}{\partial \xi_{0,\alpha}}\right)^2 \left(\frac{\partial J}{\partial \xi_{0,0}}\right)^{-2} - V(x) - U_{\mathrm{B}}\right\}\rho d^4x \tag{4.19}$$

$$\rho \equiv \rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial \xi_{0,0}}$$

It follows from (1.15) that

$$\rho U_{\mathrm{B}} = \rho U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right) = \frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho} - \frac{\hbar^2}{4m}\boldsymbol{\nabla}^2\rho \tag{4.20}$$

The last term of (4.20) has a form of divergence, and it does not contribute to dynamic equations. This term may be omitted.

If the relation (1.16)

$$\frac{\partial J}{\partial \xi_{0,0}} \neq 0 \tag{4.21}$$

takes place, the variational problems (4.15) and (4.19) are equivalent. On the contrary, if the relation (4.21) is violated we cannot be sure, that they are equivalent.

Now we introduce designation $j = \{j^0, \mathbf{j}\} = \{\rho, \mathbf{j}\} = \{j^k\}$, $k = 0, 1, 2, 3$

$$j^k = \rho_0(\boldsymbol{\xi}) \frac{\partial (x^k, \xi_1, \xi_2, \xi_3)}{\partial (x^0, x^1, x^2, x^3)} = \rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,k}}, \qquad k = 0, 1, 2, 3 \tag{4.22}$$

and add designation (4.22) to the action (4.19) by means the Lagrangian multipliers p_k, $k = 0, 1, 2, 3$. We obtain

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\xi, j, p] = \int_{V_x} \left\{ m\frac{\mathbf{j}^2}{2\rho} - V(x)\rho - \rho U_{\mathrm{B}} - p_k \left(j^k - \rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,k}} \right) \right\} d^4x, \tag{4.23}$$

$$U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right), \qquad \rho \equiv j^0$$

Note that the action (4.19) and the action (4.23) describe the same variational problem. The action (4.23) is interesting in the sense, that the Lagrangian coordinates $\xi = \{\xi_0, \boldsymbol{\xi}\}$ are concentrated in the last term of the action. The Lagrangian coordinates $\xi = \{\xi_0, \boldsymbol{\xi}\}$ are defined to within the transformation

$$\xi_0 = f_0\left(\tilde{\xi}_0\right), \qquad \xi_\alpha = f_\alpha\left(\tilde{\boldsymbol{\xi}}\right), \qquad \alpha = 1, 2, 3 \tag{4.24}$$

where f_k, $k = 0, 1, 2, 3$ are arbitrary functions. The variable ξ_0 is fictitious, and variation with respect to ξ_0 does not give an independent dynamic equation.

Variation of the action (4.23) with respect to ξ_l, $l = 0, 1, 2, 3$ leads to the dynamic equations

$$\frac{\delta \mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}}{\delta \xi_l} = -\partial_s \left(\rho_0(\boldsymbol{\xi}) p_k \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \right) + p_k \frac{\partial \rho_0}{\partial \xi_l}(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,k}} = 0, \qquad l = 0, 1, 2, 3 \tag{4.25}$$

Using identities

$$\frac{\partial J}{\partial \xi_{i,l}} \xi_{k,l} \equiv J\delta_k^i, \qquad \partial_l \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \equiv 0 \tag{4.26}$$

$$\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \equiv J^{-1} \left(\frac{\partial J}{\partial \xi_{0,k}} \frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}} \frac{\partial J}{\partial \xi_{l,k}} \right) \tag{4.27}$$

we obtain from (4.25)

$$-\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \rho_0 \partial_s p_k - \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \frac{\partial \rho_0}{\partial \xi_j} \xi_{j,s} p_k + p_k \frac{\partial \rho_0}{\partial \xi_l} \frac{\partial J}{\partial \xi_{0,k}} = 0, \qquad l = 0, 1, 2, 3$$

$$-J^{-1}\left(\frac{\partial J}{\partial \xi_{0,k}}\frac{\partial J}{\partial \xi_{l,s}}-\frac{\partial J}{\partial \xi_{0,s}}\frac{\partial J}{\partial \xi_{l,k}}\right)\rho_0\left(\boldsymbol{\xi}\right)\partial_s p_k-\left(\frac{\partial J}{\partial \xi_{0,k}}\delta_j^l-\delta_j^0\frac{\partial J}{\partial \xi_{l,k}}\right)\frac{\partial \rho_0\left(\boldsymbol{\xi}\right)}{\partial \xi_j}p_k$$
$$+p_k\frac{\partial \rho_0\left(\boldsymbol{\xi}\right)}{\partial \xi_l}\frac{\partial J}{\partial \xi_{0,k}}=0, \qquad l=0,1,2,3 \tag{4.28}$$

Simplifying (4.28) by means of the first identity (4.26), we obtain

$$J^{-1}\left(\frac{\partial J}{\partial \xi_{0,k}}\frac{\partial J}{\partial \xi_{l,s}}-\frac{\partial J}{\partial \xi_{0,s}}\frac{\partial J}{\partial \xi_{l,k}}\right)\rho_0\partial_s p_k=0 \tag{4.29}$$

Convoluting (4.29) with $\xi_{l,i}$ and using the first identity (4.26) and designations (4.22), we obtain

$$j^k\partial_i p_k-j^k\partial_k p_i=0, \qquad i=0,1,2,3 \tag{4.30}$$

Variation of (4.23) with respect to j^β gives

$$p_\beta=m\frac{j^\beta}{\rho}, \qquad \beta=1,2,3 \tag{4.31}$$

Variating (4.23) with respect to $j^0=\rho$, using designations

$$\rho_\gamma\equiv\partial_\gamma\rho, \qquad \rho_{\alpha\beta}\equiv\partial_\alpha\partial_\beta\rho$$

and taking into account relation (4.3) for $U_{\mathrm{B}}=U\left(\rho,\boldsymbol{\nabla}\rho,\boldsymbol{\nabla}^2\rho\right)$, we obtain

$$\begin{aligned}p_0 &= -\frac{m}{2\rho^2}j^\alpha j^\alpha-V\left(x\right)-\frac{\partial}{\partial\rho}\left(\rho U_{\mathrm{B}}\right)+\partial_\gamma\frac{\partial}{\partial\rho_\gamma}\left(\rho U_{\mathrm{B}}\right)-\partial_\alpha\partial_\beta\frac{\partial}{\partial\rho_{\alpha\beta}}\left(\rho U_{\mathrm{B}}\right)\\ &= -\frac{m}{2\rho^2}j^\alpha j^\alpha-V\left(x\right)-U_{\mathrm{B}}\end{aligned} \tag{4.32}$$

We note the remarkable property of the Bohm potential $U_{\mathrm{B}}=U\left(\rho,\boldsymbol{\nabla}\rho,\boldsymbol{\nabla}^2\rho\right)$, defined by the relation (4.3). The quantity p_0 is expressed via $U_{\mathrm{B}}=U\left(\rho,\boldsymbol{\nabla}\rho,\boldsymbol{\nabla}^2\rho\right)$ in such a way, as if $U\left(\rho,\boldsymbol{\nabla}\rho,\boldsymbol{\nabla}^2\rho\right)$ does not depend on ρ and its derivatives.

Eliminating p_k from the equations (4.30) by means of relations (4.31), (4.32) and setting $\mathbf{v}=\mathbf{j}/\rho$, we obtain dynamic equations in the Eulerian form (4.10).

There is another possibility. The dynamic equations (4.29) may be considered to be linear partial differential equations with respect to variables p_k. They can be solved in the form

$$p_k=b_0\left(\partial_k\varphi+g^\alpha\left(\boldsymbol{\xi}\right)\partial_k\xi_\alpha\right), \qquad k=0,1,2,3 \tag{4.33}$$

where $g^\alpha\left(\boldsymbol{\xi}\right), \quad \alpha=1,2,3$ are arbitrary functions of the argument $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$, $b_0\neq 0$ is an arbitrary real constant, and φ is the variable ξ_0, which ceases to be fictitious. Note that the constant b_0 may be eliminated, including it in the functions $\mathbf{g}=\{g^1,g^2,g^3\}$ and in the variable φ.

One can test by the direct substitution that the relation (4.33) is the general solution of linear equations (4.29). Substituting (4.33) in (4.29) and taking into account antisymmetry of the bracket in (4.29) with respect to transposition of indices k and s, we obtain

$$J^{-1}\rho_0(\boldsymbol{\xi})\left(\frac{\partial J}{\partial\xi_{0,k}}\frac{\partial J}{\partial\xi_{l,s}}-\frac{\partial J}{\partial\xi_{0,s}}\frac{\partial J}{\partial\xi_{l,k}}\right)\frac{\partial g^{\alpha}(\boldsymbol{\xi})}{\partial\xi_{\mu}}\xi_{\mu,s}\xi_{\alpha,k}=0 \tag{4.34}$$

The relation (4.34) is the valid equality, as it follows from the first identity (4.26).

Let us substitute (4.33) in the action (4.23). Taking into account the first identity (4.26) and omitting the term

$$\rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial\xi_{0,k}}\partial_k\varphi=\rho_0(\boldsymbol{\xi})\frac{\partial(\varphi,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)}$$

which does not contribute to the dynamic equations, we obtain

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\varphi,\boldsymbol{\xi},j]=\int\left\{\frac{m}{2}\frac{j^{\alpha}j^{\alpha}}{j^0}-V(x)\rho-U_{\mathrm{B}}\rho-j^k b_0\left(\partial_k\varphi+g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_{\alpha}\right)\right\}d^4x,\qquad j^0\equiv\rho \tag{4.35}$$

Variation of (4.35) with respect to $j^0\equiv\rho$ gives

$$-\frac{m\mathbf{j}^2}{2\rho^2}-V(x)-U_{\mathrm{B}}-b_0\left(\partial_0\varphi+g^{\alpha}(\boldsymbol{\xi})\partial_0\xi_{\alpha}\right)=0,\qquad U_{\mathrm{B}}=\frac{\hbar^2}{8m}\left(\frac{(\boldsymbol{\nabla}\rho)^2}{\rho^2}-2\frac{\boldsymbol{\nabla}^2\rho}{\rho}\right) \tag{4.36}$$

Variation with respect to j^{μ} gives

$$m\frac{j^{\mu}}{\rho}=b_0\left(\partial_{\mu}\varphi+g^{\alpha}(\boldsymbol{\xi})\partial_{\mu}\xi_{\alpha}\right) \tag{4.37}$$

Variation with respect to φ gives

$$\partial_k j^k=0 \tag{4.38}$$

Finally, varying (4.35) with respect to ξ_{μ} and taking into account (4.38), we obtain

$$b_0 j^k\Omega^{\alpha\mu}(\boldsymbol{\xi})\partial_k\xi_{\alpha}=0,\qquad \Omega^{a\mu}(\boldsymbol{\xi})=\left(\frac{\partial g^{\alpha}(\boldsymbol{\xi})}{\partial\xi_{\mu}}-\frac{\partial g^{\mu}(\boldsymbol{\xi})}{\partial\xi_{\alpha}}\right) \tag{4.39}$$

If

$$\det||\Omega^{\alpha\mu}||\neq 0 \tag{4.40}$$

then taking into account that the velocity $\mathbf{v}=\mathbf{j}/j^0$, one obtains from (4.39), so called Lin constraint [13]

$$\partial_0\boldsymbol{\xi}+(\mathbf{v}\boldsymbol{\nabla})\boldsymbol{\xi}=0 \tag{4.41}$$

which means that the variables $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$ are constant along the mean world lines of particles. In other words, the variables $\boldsymbol{\xi}$ are Lagrangian coordinates, which label mean world lines of particles.

However, the constraint (4.40) is not takes place always. In particular, $\Omega^{\alpha\beta} \equiv 0$ in the case of irrotational flow. Besides, the quantity $\Omega^{\alpha\beta}$ is antisymmetric, as it follows from the second relation (4.39), and

$$\det\left|\left|\Omega^{\alpha\beta}\right|\right| = \begin{vmatrix} 0 & \Omega^{12} & \Omega^{13} \\ -\Omega^{12} & 0 & \Omega^{23} \\ -\Omega^{13} & -\Omega^{23} & 0 \end{vmatrix} \equiv 0 \tag{4.42}$$

Note that identity (4.42) is a property of the three-dimensional space. In the two-dimensional space $\det\left|\left|\Omega^{\alpha\beta}\right|\right| = \left(\Omega^{12}\right)^2$. In the case of four-dimensional space we have

$$\det\left|\left|\Omega^{\alpha\beta}\right|\right| = \left(\Omega^{12}\Omega^{34} - \Omega^{13}\Omega^{24} + \Omega^{14}\Omega^{23}\right)^2$$

It seems rather strange and unexpected, that the Lin constraint (4.41) is not a corollary of the dynamic equation (4.39), although the Lin constraint (4.41) it is compatible with the dynamic equation (4.39). In the case of nonrotational flow the Euler hydrodynamic equations for perfect fluid can be obtained from the variational principle [14]. In the case of rotational flow of the same fluid the Euler hydrodynamic equations can be deduced from the variational principle, only when the Lin constraints are introduced in the action functional as side conditions, and the variables $\boldsymbol{\xi}$ are considered as dynamic variables [13]. Does it mean, that the Lagrangian coordinates $\boldsymbol{\xi}$ are inadequate dynamical variables? Maybe. It is not clear now.

From equations (4.36) - (4.41) one obtains five equations

$$-\frac{\left(\boldsymbol{\nabla}\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\alpha}\right)^2}{2m} - V\left(x\right) - U_{\mathrm{B}} - \left(\partial_0\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\partial_0\xi_{\alpha}\right) = 0, \tag{4.43}$$

$$\partial_0\boldsymbol{\xi} + \left(\mathbf{v}\boldsymbol{\nabla}\right)\boldsymbol{\xi} = 0 \tag{4.44}$$

$$\partial_0\rho + \boldsymbol{\nabla}\left(\rho\frac{\left(\boldsymbol{\nabla}\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\alpha}\right)}{m}\right) \tag{4.45}$$

for five dynamic variables $\rho, \varphi, \boldsymbol{\xi}$. Indefinite functions $\mathbf{g}\left(\boldsymbol{\xi}\right) = \left\{g^1\left(\boldsymbol{\xi}\right), g^2\left(\boldsymbol{\xi}\right), g^3\left(\boldsymbol{\xi}\right)\right\}$ are determined from initial conditions for velocity $\mathbf{v} = \mathbf{j}/\rho$. The constant b_0 is included in the indefinite functions $\varphi, \mathbf{g}\left(\boldsymbol{\xi}\right)$ The velocity $\mathbf{v}$ is expressed via dynamic variables $\rho, \varphi, \boldsymbol{\xi}$ by means of the relation

$$\mathbf{v} = \frac{\mathbf{j}}{\rho} = \frac{\left(\boldsymbol{\nabla}\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\alpha}\right)}{m} \tag{4.46}$$

5 Meaning of functions $\mathbf{g} = \left\{g^1, g^2, g^3\right\}$

The arbitrary functions $\mathbf{g} = \left\{g^1\left(\boldsymbol{\xi}\right), g^2\left(\boldsymbol{\xi}\right), g^3\left(\boldsymbol{\xi}\right)\right\}$ may be derived from initial values of hydrodynamic variables ρ, $\mathbf{v}$. Let at the initial moment $t = 0$

$$\rho\left(0, \mathbf{x}\right) = \rho_{\mathrm{in}}\left(\mathbf{x}\right), \qquad \mathbf{v}\left(0, \mathbf{x}\right) = \mathbf{v}_{\mathrm{in}}\left(\mathbf{x}\right) \tag{5.1}$$

Let us choose the initial form of labelling in the form

$$\boldsymbol{\xi}\left(0,\mathbf{x}\right)=\boldsymbol{\xi}_{\text{in}}\left(\mathbf{x}\right)=\mathbf{x},\qquad\varphi\left(0,\mathbf{x}\right)=\varphi_{\text{in}}\left(\mathbf{x}\right)=0 \tag{5.2}$$

Setting $t=0$ in (4.46), (4.18) and taking into account (5.1) and (5.2), we obtain respectively

$$\mathbf{v}\left(0,\mathbf{x}\right)=\mathbf{v}_{\text{in}}\left(\mathbf{x}\right)=\frac{\mathbf{g}\left(\mathbf{x}\right)}{m} \tag{5.3}$$

$$\rho\left(0,\mathbf{x}\right)=\rho_{\text{in}}\left(\mathbf{x}\right)=\rho_{0}\left(\mathbf{x}\right)\frac{\partial\left(\xi_{\text{in}1}\left(\mathbf{x}\right),\xi_{\text{in}2}\left(\mathbf{x}\right),\xi_{\text{in}3}\left(\mathbf{x}\right)\right)}{\partial\left(x^{1},x^{2},x^{3}\right)}=\rho_{0}\left(\mathbf{x}\right) \tag{5.4}$$

Thus, arbitrary functions $\mathbf{g}\left(\boldsymbol{\xi}\right)$ and the weight function $\rho_{0}\left(\boldsymbol{\xi}\right)$ may be uniquely determined via initial values $\rho_{\text{in}}\left(\mathbf{x}\right)$, $\mathbf{v}_{\text{in}}\left(\mathbf{x}\right)$ of quantities ρ, $\mathbf{v}$.

Eliminating functions $\mathbf{g}\left(\boldsymbol{\xi}\right)$ from dynamic equations (4.43) - (4.45) by means of relations (5.3), we obtain

$$\partial_{0}\xi_{\alpha}+\left(\frac{1}{m}\boldsymbol{\nabla}\varphi+\mathbf{v}_{\text{in}}\left(\boldsymbol{\xi}\right)\right)\boldsymbol{\nabla}\xi_{\alpha}=0,\qquad\alpha=1,2,3 \tag{5.5}$$

$$\partial_{0}\varphi+\frac{\left(\boldsymbol{\nabla}\varphi\right)^{2}}{2m}+\frac{m\left(\mathbf{v}_{\text{in}}\left(\boldsymbol{\xi}\right)\partial_{\alpha}\boldsymbol{\xi}\right)\left(\mathbf{v}_{\text{in}}\left(\boldsymbol{\xi}\right)\partial_{\alpha}\boldsymbol{\xi}\right)}{2}-mv_{\text{in}}^{\alpha}\left(\boldsymbol{\xi}\right)\mathbf{v}_{\text{in}}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\alpha}+V\left(x\right)+U_{\text{B}}\left(\rho\right)=0 \tag{5.6}$$

$$\partial_{0}\rho+\boldsymbol{\nabla}\left(\rho\left(\frac{1}{m}\boldsymbol{\nabla}\varphi+v_{\text{in}}^{\beta}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\beta}\right)\right)=0 \tag{5.7}$$

The initial values $\boldsymbol{\xi}_{\text{in}}\left(\mathbf{x}\right)$, $\varphi_{\text{in}}\left(\mathbf{x}\right)$ of hydrodynamic potentials $\boldsymbol{\xi}$, φ may be chosen universally for all flows, for instance, in the form (5.2). It means, that equations (4.44), (4.45) are essentially equations, describing the labelling evolution at fixed dynamics.

6 Description in terms of complex potential

One may form complex potential ψ from the Clebsch potentials $\boldsymbol{\xi}$, φ and the density ρ. This complex potential ψ is known as the wave function, or ψ-function. By means of a change of variables the action (4.35) can be transformed to a description in terms of a wave function [6]. Let us introduce the k-component complex function $\psi=\{\psi_{\alpha}\}$, $\alpha=1,2,...k$, defining it by the relations

$$\psi_{\alpha}=\sqrt{\rho}e^{i\varphi}u_{\alpha}(\boldsymbol{\xi}),\qquad\psi_{\alpha}^{*}=\sqrt{\rho}e^{-i\varphi}u_{\alpha}^{*}(\boldsymbol{\xi}),\qquad\alpha=1,2,...k \tag{6.1}$$

$$\psi^{*}\psi\equiv\sum_{\alpha=1}^{k}\psi_{\alpha}^{*}\psi_{\alpha} \tag{6.2}$$

where (*) means the complex conjugate, $u_\alpha(\boldsymbol{\xi})$, $\alpha = 1,2,...k$ are functions of only variables $\boldsymbol{\xi}$. They satisfy the relations

$$-\frac{i}{2}\sum_{\alpha=1}^{k}(u_\alpha^*\frac{\partial u_\alpha}{\partial \xi_\beta}-\frac{\partial u_\alpha^*}{\partial \xi_\beta}u_\alpha)=g^\beta(\boldsymbol{\xi}),\qquad \beta=1,2,...k\qquad \sum_{\alpha=1}^{k}u_\alpha^*u_\alpha=1 \tag{6.3}$$

where k is such a natural number that equations (6.3) admit a solution. In general, k depends on the form of the arbitrary functions $\mathbf{g}=\{g^\beta(\boldsymbol{\xi})\}$, $\beta=1,2,3$.

It is easy to verify, that

$$\rho=\psi^*\psi,\qquad j^\mu=-\frac{ib_0}{2m}(\psi^*\partial_\mu\psi-\partial_\mu\psi^*\cdot\psi),\qquad \mu=1,2,3 \tag{6.4}$$

The variational problem with the action (4.35) appears to be equivalent [6] to the variational problem with the action functional

$$\begin{aligned}\mathcal{A}[\psi,\psi^*] &= \int\left\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\psi)+\frac{b_0^2(\psi^*\boldsymbol{\nabla}\psi-\boldsymbol{\nabla}\psi^*\cdot\psi)^2}{8m\psi^*\psi}\right.\\ &\quad\left.-\frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}(\psi^*\psi))^2}{\psi^*\psi}-V(x)\,\psi^*\psi\right\}\mathrm{d}^4x\end{aligned} \tag{6.5}$$

where $\boldsymbol{\nabla}=\{\partial_\alpha\}$, $\alpha=1,2,3.$

Let us consider the case, when the number k of the wave function components is equal to 2. In this case the wave function $\psi=\left\{\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right\}$ has four real components. The number of hydrodynamic variables ρ, $\mathbf{j}$ is also four, and we may hope that the first three equations (6.3) can be solved for any choice of functions $\mathbf{g}$. For the two-component wave function ψ we have the identity

$$(\psi^*\boldsymbol{\nabla}\psi-\boldsymbol{\nabla}\psi^*\cdot\psi)^2\equiv-4\rho\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi+(\boldsymbol{\nabla}\rho)^2+4\rho^2\sum_{\alpha=1}^{3}(\boldsymbol{\nabla}s_\alpha)^2 \tag{6.6}$$

where

$$\rho=\psi^*\psi,\qquad s_\alpha=\frac{\psi^*\sigma_\alpha\psi}{\rho},\qquad \alpha=1,2,3 \tag{6.7}$$

σ_α are 2×2 Pauli matrices

$$\sigma_1=\left(\begin{array}{cc}0&1\\1&0\end{array}\right),\qquad \sigma_2=\left(\begin{array}{cc}0&-i\\i&0\end{array}\right),\qquad \sigma_3=\left(\begin{array}{cc}1&0\\0&-1\end{array}\right), \tag{6.8}$$

Substituting (6.6) in (6.5), we obtain

$$\begin{aligned}\mathcal{A}[\psi,\psi^*] &= \int\left\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{b_0^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi+\frac{b_0^2}{8m}\rho\,(\boldsymbol{\nabla}s_\alpha)^2\right.\\ &\quad\left.+\frac{b_0^2}{8m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho}-\frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho}-V(x)\,\psi^*\psi\right\}\mathrm{d}^4x\end{aligned} \tag{6.9}$$

If we choose the arbitrary constant b_0 in the form $b_0 = \hbar$, the action (6.9) takes the form

$$\mathcal{A}[\psi,\psi^*] = \int \left\{ \frac{i\hbar}{2}(\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi) - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi + \frac{\hbar^2}{8m}\rho\boldsymbol{\nabla}s_\alpha\boldsymbol{\nabla}s_\alpha - V(x)\,\rho \right\} \mathrm{d}^4x \qquad (6.10)$$

In the case, when the wave function ψ is one-component, for instance $\psi = \left\{\begin{smallmatrix}\psi_1\\0\end{smallmatrix}\right\}$, the quantities $\mathbf{s} = \{s_1, s_2, s_3\}$ are constant ($s_1 = 0,\quad s_2 = 0,\quad s_3 = 1$), the action (6.10) turns into

$$\mathcal{A}[\psi,\psi^*] = \int \left\{ \frac{i\hbar}{2}(\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi) - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi - V(x)\,\psi^*\psi \right\} \mathrm{d}^4x \qquad (6.11)$$

The dynamic equation, generated by the action (6.11), is the Schrödinger equation

$$i\hbar\partial_0\psi + \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi - V(x)\,\psi = 0 \qquad (6.12)$$

This dynamic equation describes the flow of the fluid.

In the general case the dynamic equation, generated by the action (6.9) has the form

$$ib_0\partial_0\psi + \frac{b_0^2}{2m}\boldsymbol{\nabla}^2\psi + \frac{b_0^2}{8m}\boldsymbol{\nabla}^2 s_\alpha\cdot(s_\alpha - 2\sigma_\alpha)\,\psi - \frac{b_0^2}{4m}\frac{\boldsymbol{\nabla}\rho}{\rho}\boldsymbol{\nabla}s_\alpha\sigma_\alpha\psi - \left(1 - \frac{b_0^2}{\hbar^2}\right)U_{\mathrm{B}}\psi - V(x)\,\psi = 0 \qquad (6.13)$$

where U_{B} is determined by the relation (4.3). Deriving dynamic equation (6.13), we have used the identities

$$\mathbf{s}^2 \equiv 1, \qquad s_\alpha\boldsymbol{\nabla}s_\alpha \equiv 0, \qquad \boldsymbol{\nabla}s_\alpha\left(\boldsymbol{\nabla}s_\alpha\right) + s_\alpha\boldsymbol{\nabla}^2 s_\alpha \equiv 0$$

In the case if $b_0 = \hbar$ the equation (6.13) turns into

$$i\hbar\partial_0\psi + \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi - V(x)\,\psi + \frac{\hbar^2}{8m}\boldsymbol{\nabla}^2 s_\alpha\cdot(s_\alpha - 2\sigma_\alpha)\,\psi - \frac{\hbar^2}{4m}\frac{\boldsymbol{\nabla}\rho}{\rho}\boldsymbol{\nabla}s_\alpha\sigma_\alpha\psi = 0 \qquad (6.14)$$

where two last terms differ this equation from the Schrödinger equation. These two terms are responsible for vorticity of the flow. In accordance with the Schrödinger equation the particle spin is an attribute of the quantum particle, and it does not influence on the flow of the statistical ensemble. According to the equation (6.14) a pointlike spin-free particle spin may have a spin, generated by the vorticity of the statistical ensemble flow.

Using the change of variables (6.1), (6.3), we did not use the fact, that the solution of equations (4.39) is a solution of the equations (4.41). In the case of description in terms of the wave function ψ we have not the problem, which we have at description in terms of the generalized stream function $\boldsymbol{\xi}$, when there are such solutions of (4.39), which are not solutions of (4.41).

7 Concluding remarks

In this paper we try to construct the uniform formalism for description of physical (stochastic and deterministic) systems. We use the statistical ensemble as a basic object of dynamics, using the fact that the statistical ensemble is a continuous dynamic system independently of whether its elements are stochastic or dynamic systems. Such an approach admits one to describe quantum systems, considering them as stochastic system and using *only* principles of classical (not quantum) physics at this description. Besides, the developed technique may be applied for description of classical inviscid fluids.

Existence of quantum particles together with the uniform description of classical and quantum particles generates an alternative to quantum nature of particles in microcosm. Indeed, the quantum paradigm supposes description of particle motion by means of principles of quantum theory. The quantum principles suppose a description in terms of linear dynamic equations for a wave function, which is introduced axiomatically. In the model conception, when the wave function is simply a way of description of an ideal continuous media (statistical ensemble), the dynamic equations appear to be linear only for nonrotational flow of the medium. In the case of a rotational flow the Schrödinger equation ceases to be linear differential equation. Nonlinear terms appear in the equation (6.14), describing a statistical ensemble of quantum particles.

The uniform method of the particle motion description admits one to refuse the quantum principles as needless ones. However, it put the question: "What is the nature of the particle motion stochasticity?" There is the only answer. The multivariance (stochasticity) of the particle motion is conditioned by the properties of the space-time geometry. In the twentieth century, when multivariant geometries were unknown, such an approach was impossible and a use of the geometric paradigm was impossible. But now, when multivariant geometries are known, the geometrical paradigm, which denies the quantum principles as the prime principles, looks more natural, than the quantum paradigm, based on needless quantum principles, because a change of the space-time geometry looks more reasonable, than a change of the dynamics principles..

We considered four different methods of the statistical ensemble description. Consideration of deterministic, stochastic and quantum systems as special cases of a physical system is conditioned be the fact, that the space-time geometry may be non-Riemannian, and the motion of particle may be multivariant (stochastic) primordially [15].

Constructing uniform formalism, we did not introduce any new hypotheses. We worked with physical principles (not with single physical phenomena). We realized the logical reloading [16], i.e. replacement of basic concepts of a theory. A single particle as a basic concept of the particle dynamics has been replaced by another basic concept: statistical ensemble of dynamic (or stochastic) particles. The logical reloading is a logical operation, which is used rare in the theoretical physics.

Usually the statistical description is used for a description of particles, when information on the particle dynamics is incomplete. This incompleteness may be connected with indefinite initial conditions or with stochasticity of the particle motion. Usually the statistical description is introduced as some external operation with respect to the particle dynamics. The logical reloading admits one to introduce the statistical description into the particle dynamics. The statistical description becomes to be an internal dynamical operation, which does not use the concept of probability. Dynamical conception of statistical description, when one considers many identical independent particles, but not a probability of a state of a single particle, extends its capacities, because the probabilistic description is only a special case of the statistical description.

The logical reloading turns the statistical description into a component of the particle dynamics. This circumstance extends capacity of the particles dynamics. In particular, in the relativistical case a description of the pair production (and annihilation) becomes to be possible in the framework of the uniform formalism of the particle dynamics.

References

[1] Yu. A. Rylov, Non-Riemannian model of space-time responsible for quantum effects. *J. Math. Phys.* **32**, 2092-2098, (1991).

[2] Yu. A. Rylov, Extremal properties of Synge's world function and discrete geometry. *J. Math. Phys.* **31**, 2876-2890, (1990).

[3] Yu. A. Rylov, Geometry without topology as a new conception of geometry. *Int. J. Math. Math. Sci.*, **30**, 733-760, (2002).

[4] Yu. A. Rylov, Tubular geometry construction as a reason for new revision of the space-time conception. *e-print /physics/0504031*

[5] J.L. Synge, *Relativity: The General Theory*, North-Holland, Amsterdam, 1960.

[6] Yu. A. Rylov, Spin and wave function as attributes of ideal fluid. *J. Math. Phys.*, **40,** 256-278, (1999).

[7] D. Bohm, On interpretation of quantum mechanics on the basis of the "hidden" variable conception. *Phys.Rev.* **85,** 166, 180, (1952).

[8] Yu.A. Rylov, Dynamics of stochastic systems and pecularities of measurements in them. *e-print 0210003*

[9] Yu.A.Rylov, Classical description of pair production. *e-print, physics/0301020* .

[10] A. Clebsch, Über eine allgemaine Transformation der hydrodynamischen Gleichungen, *J. reine angew. Math.* **54** , 293-312 (1857).

[11] A. Clebsch, Ueber die Integration der hydrodynamischen Gleichungen, *J. reine angew. Math.* **56** , 1-10, (1859).

[12] Yu. A. Rylov, Hydrodynamic equations for incompressible inviscid fluid in terms of generalized stream function . *Int. J. Math. & Mat. Sci.* **2004**, *No. 11, 21, pp. 541-570.*

[13] Lin, C.C. Hydrodynamics of Helium II. *Proc. Int. Sch Phys.* Course XXI, pp. 93-146, New York, Academic, 1963.

[14] Davydov, B. Variational principle and canonical equations for perfect fluid, *Doklady Akadedimii Nauk USSR*, **69**, 165-168, (1949), (in Russian)

[15] Yu. A. Rylov, Necessity of the general relativity revision and free motion of particles in non-Riemannian space-time geometry *e-print 1001.5362v1*

[16] Yu. A.Rylov, Logical reloading in statistical description of particle dynamics. *e-print 1006.1254v1 .*