Relativistic nature of nonrelativistic quantum mechanics and multivariance of the space-time geometry

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Abstract

One uses the investigation strategy: "Find a mistake in foundation of physics and correct it". Two fundamental defects are corrected: (1) nonrelativistic concept of the particle state, (2) inadequacy of the differential geometry formalism at the metric approach to geometry. Usage of relativistic concept of the particle state admits one to create united formalism for description of deterministic and stochastic particles motion. The quantum mechanics is founded as a statistical description of the stochastic particles motion. Stochasticity of the quantum particles motion is explained by multivariance of the space-time geometry, which leads to the world lines wobbling of elementary particles. Relativistic concept of the particle state and the metric approach to the space-time geometry admit one to construct the skeleton conception of elementary particles. The skeleton conception admits one to investigate the arrangement of elementary particles (but not only to systamatize them, ascribing quantum numbers to them).

Key words: relativistic particle state; united formalism of dynamics; metric approach to geometry; multivariant geometry; inadequacy of differential geometry formalism

1 Introduction

We believe that many problems of the contemporary theoretical physics are results of some mistakes in the physics foundation. Finding these mistakes and correcting them, one can solve many problems of theoretical physics, if these mistakes were made in the foundations of physics on a deep level. Correction of such mistakes (if they exist) may change essentially the direction of physics development. It can lead to development of a new formalism (of course, if the mistake does exist and concerns foundations of physics).

In this paper we consider two essential mistakes in the physics foundation. The first mistake is connected with the fact that the relativistic particle dynamics is not completely relativistic in the sense that dynamic equations of the particle motion are relativistic, whereas the particle state is described nonrelativistically. It is unessential at description of deterministic particles, but it is essential at description of stochastic (quantum) particles. Introduction of relativistical state of a particle admits one to create a united formalism for description of deterministic and stochastic particles. As a result one succeeded to explain quantum effects as a result of a statistical description of the stochastic particle motion, and the quantum principles cease to be primary principles of nature.

The second mistake is connected with the concept of multivariant geometry. Conventional space-time geometry does not acknowledge concept of multivariance in a geometry. Consideration of multivariant space-time geometries admits one to explain stochastic motion of free quantum particles. Consideration of multivariant space-time geometries admits one to construct skeleton conception of elementary particles. The skeleton conception admits one to investigate arrangement of elementary particles, whereas the conventional elementary particle theory can only systematize elementary particles, ascribing them quantum numbers.

The relativistic particle dynamics was not formulated completely in the beginning of the twentieth century. Dynamic equations of the particle motion were relativistic, but the state of a particle remains to be nonrelativistic. It is unessential for dynamics of deterministic particles. But it is essential for description of nondeterministic (stochastic) particles. Without a use of relativistic description of the particle state one could not describe a stochastic motion of quantum particles in terms of classical particle dynamics. As a result one was forced to invent an axiomatic conception, known as nonrelativistic quantum mechanics. In reality nonrelativistic quantum mechanics is a relativistic conception in the sense, that only the mean motion of nonrelativistic quantum particles is nonrelativistic, whereas the stochastic component of the quantum particle motion is relativistic. The mean value of the stochastic component vanishes at averaging, and it is not taken into account in the quantum mechanics. Nevertheless for construction of the quantum particle dynamics one needs to use the relativistic conception of the particle dynamics. A true (relativistic) description of the particle state admits one to describe motion of quantum particles as a statistical description of the stochastic classical particles motion. As a result one obtains a statistical foundation of quantum mechanics, where principles of quantum mechanics are secondary principles, which are obtained as a result of a statistical description of the classical stochastic particles dynamics.

Besides, for statistical foundation of the quantum mechanics one needs to explain, what is the wave function and how it appears in the statistical description.

It appears, that the wave function is a method of the ideal continuous medium description [1]. Indeed, a statistical description of stochastic particles is a description of a statistical ensemble, which is a set of many independent identical particles (deterministic or stochastic). Motion of a statistical ensemble is close to a motion of a gas, which is a set of many identical particles, interacting via collisions. Methods of a gas description coincide with methods of the statistical ensemble description. This circumstance explains appearance of the wave function in quantum mechanics, provided the quantum mechanics is a statistical description of the stochastic particle motion in terms of the statistical ensemble.

After explanation of the quantum mechanics as a statistical description of stochastically moving particles the question arose, why free particles of small mass move stochastically. The reason of the particle stochasticity appeared to be a discreteness of the space-time geometry. More exactly, the reason of the stochastic motion is a multivariance of the discrete space-time geometry.

Multivariance is such a property of a geometry, when at a point P there are many vectors $\mathbf{PQ}$, $\mathbf{PQ}'$, $\mathbf{PQ}''$,... which are equivalent to the vector $\mathbf{AB}$ at the point A, but vectors $\mathbf{PQ}$, $\mathbf{PQ}'$, $\mathbf{PQ}''$,... are not equivalent between themselves. Even in the geometry of Minkowski the equivalence of spacelike vectors is multivariant. For instance, vector $\mathbf{PQ} = (r_1, r_1 \cos\phi_1, r_1 \sin\phi_1, z)$ and vector $\mathbf{PQ}' = (r_2, r_2 \cos\phi_2, r_2 \sin\phi_2, z)$ are equivalent to the spacelike vector $\mathbf{AB} = (0, 0, 0, z)$ for arbitrary values of quantities r_1, ϕ_1, r_2, ϕ_2, but vectors $\mathbf{PQ}$ and $\mathbf{PQ}'$ are not equivalent, generally speaking. Equivalence of timelike vectors is single-variant in the geometry of Minkowski $\mathcal{G}_{\mathrm{M}}$, but it is multivariant in the discrete geometry $\mathcal{G}_{\mathrm{d}}$. Multivariance of a vectors which are tangent to the particle world line leads to wobbling of the world line, which means a stochasticity of the particle.

Equivalence of two vectors $\mathbf{PQ}$ and $\mathbf{AB}$ is defined as follows

$$\mathbf{PQ}\mathrm{eqv}\mathbf{AB}: \quad (\mathbf{PQ}.\mathbf{AB}) = |\mathbf{PQ}| \cdot |\mathbf{AB}| \wedge |\mathbf{PQ}| = |\mathbf{AB}| \tag{1.1}$$

where $(\mathbf{PQ}.\mathbf{AB})$ is the scalar product of vectors $\mathbf{PQ}$ and $\mathbf{AB}$, and $|\mathbf{PQ}|$ is the length of the vector $\mathbf{PQ}$.

Note that the discrete geometry $\mathcal{G}_{\mathrm{d}}$ is such a geometry, where distance $\rho(A, B)$ between any points A and B is greater than the elementary length λ_0

$$|\rho(A, B)| \notin (0, \lambda_0), \quad \forall A, B \in \Omega \tag{1.2}$$

Here Ω is the set of points, where the geometry is given. Let us stress that the relation (1.2) is a restriction on the form of the distance function ρ, but not on the point set Ω. In particular, the set Ω may coincide with the manifold Ω_{M}, where the geometry of Minkowski is given. Instead of the distance function ρ it is more convenient to use the world function $\sigma = \frac{1}{2}\rho^2$, because world function is always real, even in the geometry of Minkowski, where ρ is imaginary for spacelike vectors. World function σ_{d} of the discrete geometry $\mathcal{G}_{\mathrm{d}}$ may have the form

$$\sigma_{\mathrm{d}}(P, Q) = \sigma_{\mathrm{M}}(P, Q) + \frac{\lambda_0^2}{2}\mathrm{sgn}(\sigma_{\mathrm{M}}(P, Q)) \tag{1.3}$$

where σ_{M} is the world function of geometry of Minkowski. In the inertial coordinate system σ_{M} has the form

$$\sigma_{\mathrm{M}}(x,x') = \frac{1}{2}g_{ik}\left(x^i - x'^i\right)\left(x^k - x'^k\right), \quad g_{ik} = \mathrm{diag}\left(c^2,-1,-1,-1\right) \tag{1.4}$$

Here and later on there is a summation $(0 \div 3)$ over repeating Latin indices and $(1 \div 3)$ on the Greek ones. It is easy to verify that (1.3) satisfies the restriction (1.2).

In terms of the world function the scalar product $(\mathbf{PQ}.\mathbf{AB})$ and the length $|\mathbf{PQ}|$ may be presented in the form

$$(\mathbf{PQ}.\mathbf{AB}) = \sigma(P,B) + \sigma(Q,A) - \sigma(P,A) - \sigma(Q,B) \tag{1.5}$$

$$|\mathbf{PQ}| = \sqrt{2\sigma(P,Q)} \tag{1.6}$$

The coordinateless formulas (1.5) and (1.6) are valid in any physical geometry. The physical geometry is such a geometry, which can be described completely in terms and only terms of the world function σ. Description of a geometry in terms of the world function σ (or in terms of the distance function ρ) is known as a metric approach to geometry.

Unfortunately, the metric approach to geometry cannot be realized consistently at the conventional approach to geometry. The reason of this circumstance is connected with inadequacy of the linear vector space operations at the metric approach to geometry.

Metric approach to geometry means that a geometry is described in coordinateless way in terms of the distance function ρ or world function $\sigma = \frac{1}{2}\rho^2$. Metric approach is necessary for recognition of the same geometrical object in different geometries. Geometric vector (g-vector) $\mathbf{PQ} = \{P,Q\}$ is the ordered set of two points $P,Q \in \Omega$. In a geometry $\mathcal{G} = \{\sigma,\Omega\}$ there are many g-vectors $\mathbf{AB}$, which are equivalent to g-vector $\mathbf{PQ}$. They defined by the relation (1.1)

If the equivalence relation is transitive, and for any g-vectors it follows from $(\mathbf{P}_1\mathbf{Q}_1\mathrm{eqv}\mathbf{AB}) \wedge (\mathbf{P}_2\mathbf{Q}_2\mathrm{eqv}\mathbf{AB})$, that $(\mathbf{P}_1\mathbf{Q}_1\mathrm{eqv}\mathbf{P}_2\mathbf{Q}_2)$, then the set of all g-vectors $\mathbf{PQ}$, $(\mathbf{PQ}\mathrm{eqv}\mathbf{AB})$ forms the equivalence class $[\mathbf{AB}]$ of the g-vector $\mathbf{AB}$. In this case the geometry $\mathcal{G} = \{\sigma,\Omega\}$ is single-variant. If the equivalence relation is intransitive, the geometry $\mathcal{G} = \{\sigma,\Omega\}$ is multivariant.

In the single-variant geometry $\mathcal{G} = \{\sigma,\Omega\}$ one can connect any equivalence class $[\mathbf{AB}]$ with a linear vector (linvector) $u \in \mathcal{L}_n$ of a linear vector space $\mathcal{L}_n$ and apply operations defined for linvectors in $\mathcal{L}_n$ for equivalence classes of geometry $\mathcal{G} = \{\sigma,\Omega\}$.

In the multivariant geometry $\mathcal{G} = \{\sigma,\Omega\}$ one cannot to define linear operation of the linear vector space $\mathcal{L}_n$. If nevertheless one defines a summation of two g-vectors and multiplication of a g-vector by a number, these operations appear to be many-valued, because such a definition of these operations contains the relation of equivalence (1.1), which is many-valued in the multivariant geometry $\mathcal{G} = \{\sigma,\Omega\}$.

The geometry of Minkowski is single-variant with respect to timelike vectors and it is multivariant with respect to spacelike vectors. Usually one does not consider spacelike vectors and spacelike world lines, supposing, that tachyons do not exist. Consideration of tachyons in geometry of Minkowski shows that world lines of tachyons wobble with infinite amplitude, and a single tachyon cannot be detected. However, the tachyon gas can be detected by its gravitational field. As a result the tachyons may exist, and the tachyon gas is the best candidate for the dark matter [2].

In the contemporary geometry one does not distinguish between linvectors and g-vectors, supposing that the g-vectors forms the equivalence classes. It is supposed that the equivalence relation is transitive in any geometry, and one supposes that there are no geometries with intransitive equivalence relation. If intransitive equivalence relation appears accidentally (for instance, in Riemannian geometry for g-vectors having different origins), then such an equivalence is restricted by some additional condition (parallel transport), or the equivalence relation is ignored as in the case of spacelike vectors in the geometry of Minkowski. Such a denial of multivariance of the g-vectors equivalence relation is a mistake, which can lead to erroneous conclusions.

In the presented paper we overcome two defects of the contemporary elementary particles theory. The two defects are connected with a use of nonrelativistic concept of the particle state and with disregard of the concept of multivariance in the space-time geometry. Correction of these defects admits one to construct the skeleton conception of elementary particles [3], which allows one to determine structure of elementary particles (but not only to systematize the elementary particles and to ascribe some quantum numbers to any elementary particle).

2 Relativistic concept of the particle state

After explanation of heat phenomena by means of the kinetic gas theory it was reasonable to think, that quantum effects may be explained as some stochastic motion of microparticles. Some researchers [4, 5] tried to obtain quantum mechanics as a statistical description of stochastically moving microparticles. They failed to explain the quantum mechanics as a statistical description of stochastically moving particles. Moyal [4] tried to reduce quantum dynamic equations to the form, which is characteristic for dynamic equations of stochastic processes. Fenyes [5] tried to obtain statistical description, using similarity between the Schrödinger equation and the Fokker equation for diffusion processes. Both authors used the concept of the wave function without understanding, what it means. Explanation of quantum phenomena is hardly possible without understanding, what is the wave function. However, in the beginning of the twentieth century nobody knew, what is the wave function.

The fact, that the Schrödinger equation may be reduced to a description of an irrotational flow of some quantum fluid, was shown by Madelung [6]. However, representation of the hydrodynamic equations for ideal fluid in terms of the wave function needs a complete integration of hydrodynamic equations.

For transition from the Schrödinger equation to the system of four hydrodynamic equations, the complex Schrödinger equation for the wave function $\psi = \sqrt{\rho}\exp(i\varphi/\hbar)$ is represented in the form of two real equations for amplitude $\sqrt{\rho}$ and for the phase φ. To obtain hydrodynamic equations, it is sufficient to take gradient from the equation for the phase φ. As a result one obtains four dynamic equations, which turn into hydrodynamic equations after introducing proper designations. In other words, for transition from dynamic equations in terms of the wave function to the hydrodynamic form of these equations, one needs to differentiate dynamic equations. On the contrary, to pass from hydrodynamic form of dynamic equations to their representation in terms of the wave function, one needs to integrate dynamic equations. In the case of the irrotational flow this integration is carried out rather simply, whereas in the case of vortical flow the way of integration became to be known only in the end of twentieth century [1].

Bohm [7] used the hydrodynamic representation of the Schrödinger equation for interpretation of quantum mechanics. He started from the wave function and quantum principles and interpreted them in hydrodynamic terms. However, he could not found quantum mechanics on the basis of hydrodynamics, because for such a foundation he would start from hydrodynamic concepts and equations, in order to obtain the wave function in hydrodynamic terms. He could not make this, because in this case he would be forced to integrate hydrodynamic equations in general case, but not only for irrotational flows. Integration of the hydrodynamic equations was not known almost during the whole twentieth century. It was rather complicated mathematical problem.

Information on other attempts of a statistical foundation of quantum mechanics can be found in the book by Holland [8]. All authors tried to found the nonrelativistic quantum phenomena on the basis of nonrelativistic statistical description. This circumstance was the main reason of failures. Besides, they did not understand, that the wave function is a method of the continuous medium description. The nonrelativistic quantum mechanics describes a mean motion of particles, and the mean motion is nonrelativistic. However, the nonrelativistic character of the mean motion does not mean, that the exact particle motion is also nonrelativistic. Stochastic component of the particle motion may be relativistic, and this component disappear at the averaging. To obtain a correct description one should use a relativistic statistical description.

In the nonrelativistic physics the state of the particle is described as a point in the phase space of coordinates and momenta. In the relativistic physics the particle state is described as a world line in the space-time. In the nonrelativistic physics the state density is defined as factor n in the relation

$$dN = ndV \tag{2.1}$$

where dN is the number of particles in the volume dV of the phase space. the quantity n is nonnegative. It turns to probability density w at a proper normalization.

In the relativistic physics the particle state density j^k is defined as a factor in the relation

$$dF = j^k dS_k \tag{2.2}$$

where dF is the flux of world lines through three-dimensional spacelike area dS_k. As far as the state of a statistical ensemble is a state density of particles, the state density appears to be different in the relativistic physics and in the nonrelativistic one. Nonrelativistic state density w leads to the probabilistic conception of the statistical ensemble, whereas the relativistic state density j^k leads to the dynamic (hydrodynamic) conception of the statistical ensemble [9, 10, 11].

Hydrodynamic equations have the form

$$\partial_0 \rho + \boldsymbol{\nabla} (\rho \mathbf{v}) = 0, \quad \partial_0 \mathbf{v} + (\mathbf{v} \boldsymbol{\nabla}) \mathbf{v} = -\frac{1}{\rho} \boldsymbol{\nabla} p \tag{2.3}$$

where $\rho, \mathbf{v}, p = p(\rho)$ are respectively density, velocity and pressure in the fluid. The system of four hydrodynamic equations do not form a complete system of dynamic equations of the fluid. It forms only a closed subsystem of the complete system of dynamic equations. To obtain the complete system of dynamic equations of the fluid, one needs to add the dynamic equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(t, \mathbf{x}) \tag{2.4}$$

describing motion of the fluid particles in the given field of velocities. Formal difference between the system (2.3) and the system (2.3), (2.4) is as follows. The complete system of equations (2.3), (2.4) can be derived from the variational principle, whereas the closed subsystem (2.3) cannot be derived from the variational principle in general case. It can be obtained only in the case of irrotational motion [12].

Equations (2.4) are ordinary differential equations, whereas equations (2.3) are partial differential equations. From viewpoint of hydrodynamicists solution of equations (2.3) is the most difficult and important problem of hydrodynamics. If the equations (2.3) have been solved and the velocity field $\mathbf{v}(t, \mathbf{x})$ has been determined, a solution of equations (2.4) seems a very simple problem as compared with a solution of (2.3). As a result hydrodynamicists are apt to consider the closed subsystem (2.3) as a system of hydrodynamic equations, ignoring equations (2.4). To obtain description of hydrodynamic equations in terms of the wave function, one needs to integrate equations (2.3), (2.4).

Let us rewrite equations (2.4) in the form

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + (\mathbf{v} \boldsymbol{\nabla}) \boldsymbol{\xi} = 0 \tag{2.5}$$

where $\boldsymbol{\xi}(t, \mathbf{x}) = \boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ are independent integrals of equations (2.4). Equations (2.4) and (2.5) are equivalent, and variables $\boldsymbol{\xi}$ may be considered as Lagrangian coordinates, labelling the fluid particles, because according to (2.5) the variable $\boldsymbol{\xi}$ is constant along the world line of the fluid particle. Formally equations (2.5) are partial differential equations, but they may be reduced to the form of ordinary differential equations (2.4).

Let us consider the action for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ of free stochastic particles $\mathcal{S}_{\mathrm{st}}$, whose mean motion is nonrelativistic.

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x},\mathbf{u}] = \int\int_{V_\xi} \left\{\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}\rho_0(\boldsymbol{\xi})\,dtd\boldsymbol{\xi}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \tag{2.6}$$

The variable $\mathbf{x} = \mathbf{x}(t,\boldsymbol{\xi})$ describes the regular component of the particle motion. The variable $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$ describes the mean value of the stochastic velocity component, $\hbar$ is the quantum constant. The second term in (2.6) describes the kinetic energy of the stochastic velocity component. The third term describes interaction between the stochastic component $\mathbf{u}(t,\mathbf{x})$ and the regular component $\dot{\mathbf{x}}(t,\boldsymbol{\xi})$. The operator

$$\boldsymbol{\nabla} = \left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\} \tag{2.7}$$

is defined in the space of coordinates $\mathbf{x}$. The quantity $\rho_0(\boldsymbol{\xi})$ is the weight function.

Formally the action (2.6) looks as an action for a set of deterministic particles, interacting via some force field $\mathbf{u}$. Variation of (2.6) with respect to $\mathbf{u}$ gives

$$\mathbf{u} = \mathbf{u}(t,\mathbf{x}) = -\frac{\hbar}{2m}\boldsymbol{\nabla}\ln\rho, \tag{2.8}$$

where

$$\rho = \rho_0(\boldsymbol{\xi})\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x^1,x^2,x^3)} = \rho_0(\boldsymbol{\xi})\left(\frac{\partial(x^1,x^2,x^3)}{\partial(\xi_1,\xi_2,\xi_3)}\right)^{-1} \tag{2.9}$$

Variation with respect $\mathbf{x}$ gives

$$\delta\mathbf{x}: \qquad -m\frac{d^2\mathbf{x}}{dt^2} + \boldsymbol{\nabla}\left(\frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right) = 0 \tag{2.10}$$

Substituting (2.8) in (2.10) and considering ρ as a function of $t,\mathbf{x}$, one obtains

$$m\frac{d^2\mathbf{x}}{dt^2} = -\boldsymbol{\nabla}U_{\mathrm{B}} \tag{2.11}$$

where d/dt means the substantial derivative with respect to time t

$$\frac{dF}{dt} \equiv \frac{\partial(F,\xi_1,\xi_2,\xi_3)}{\partial(t,\xi_1,\xi_2,\xi_3)}$$

$\boldsymbol{\nabla}$ is gradient in the space of coordinates x, and U_{B} is so-called Bohm potential

$$\begin{aligned} U_{\mathrm{B}}(t,\mathbf{x}) &= -\frac{m}{2}\mathbf{u}^2 + \frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u} = U\left(\rho,\boldsymbol{\nabla}\rho,\boldsymbol{\nabla}^2\rho\right) \\ &= \frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho^2} - \frac{\hbar^2}{4m}\frac{\boldsymbol{\nabla}^2\rho}{\rho} = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}^2\sqrt{\rho} \end{aligned} \tag{2.12}$$

Let us transform the dynamic equation (2.11) to the form, where variables $\boldsymbol{\xi}$ are dependent dynamic variables, and $t, \mathbf{x}$ are independent dynamic variables. At this representation the variables $\boldsymbol{\xi}$ (Clebsch potentials [14, 15]) may be considered as a generalized stream function, because they have the property of the stream function: (1) they label the world lines of the fluid particles and (2) some combination of the derivatives of $\boldsymbol{\xi}$ satisfy the continuity equation identically at any values of $\boldsymbol{\xi}$. (See for details [16]).

The dynamic equation (2.11) can be obtained from the action

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x}] = \int\int_{V_{\boldsymbol{\xi}}} \left\{ \frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 - U_{\mathrm{B}}(t,\mathbf{x}) \right\} \rho_0(\boldsymbol{\xi})\, dt d\boldsymbol{\xi} \tag{2.13}$$

where $\mathbf{x} \equiv \mathbf{x}(t, \boldsymbol{\xi})$. The variables ρ and $U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right)$ are defined by the relations (2.12), (2.9).

To transform the action (2.13) to independent variables $x = \left\{x^k\right\} = \{t, \mathbf{x}\}$, we use the parametric representation of the mean world lines $\mathbf{x} \equiv \mathbf{x}(t, \boldsymbol{\xi})$. Let

$$x^k = x^k(\xi_0, \boldsymbol{\xi}) = x^k(\xi), \qquad k = 0, 1, 2, 3 \tag{2.14}$$

where $\xi = \{\xi_k\} = \{\xi_0, \boldsymbol{\xi}\}$, $k = 0, 1, 2, 3$. The shape of the world line is described by x^k, considered as a function of ξ_0 at fixed $\boldsymbol{\xi}$. The action (2.13) can be rewritten in the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[x] = \int_{V_{\boldsymbol{\xi}}} \left\{ \frac{m}{2}\left(\frac{\partial\mathbf{x}}{\partial\xi_0}\right)^2 \left(\frac{\partial x^0}{\partial\xi_0}\right)^{-1} - U_{\mathrm{B}}\frac{\partial x^0}{\partial\xi_0} \right\} \rho_0(\boldsymbol{\xi})\, d^4\xi, \tag{2.15}$$

Let us consider the variables $\xi = \{\xi_k\}$, $k = 0, 1, 2, 3$ as dependent variables and variables $x = \left\{x^k\right\}$ as independent ones. We consider the Jacobian

$$J = \frac{\partial(\xi_0,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)} = \det\left|\left|\xi_{l,k}\right|\right|, \qquad \xi_{l,k} \equiv \frac{\partial\xi_l}{\partial x^k} \qquad l, k = 0, 1, 2, 3 \tag{2.16}$$

as a four-linear function of variables $\xi_{l,k} \equiv \partial_k\xi_l$, $l, k = 0, 1, 2, 3$. We take into account that

$$\frac{\partial x^k}{\partial\xi_0} = \frac{\partial\left(x^k,\xi_1,\xi_2,\xi_3\right)}{\partial(\xi_0,\xi_1,\xi_2,\xi_3)} = \frac{\partial\left(x^k,\xi_1,\xi_2,\xi_3\right)}{\partial(x^0,x^1,x^2,x^3)}\frac{\partial(x^0,x^1,x^2,x^3)}{\partial(\xi_0,\xi_1,\xi_2,\xi_3)} = J^{-1}\frac{\partial J}{\partial\xi_{0,k}} \tag{2.17}$$

and

$$\rho = \rho_0(\boldsymbol{\xi})\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x^1,x^2,x^3)} = \rho_0(\boldsymbol{\xi})\frac{\partial(x^0,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)} = \rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial\xi_{0,0}} \tag{2.18}$$

The action (2.15) takes the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\xi] = \int_{V_x} \left\{ \frac{m}{2}\left(\frac{\partial J}{\partial\xi_{0,\alpha}}\right)^2 \left(\frac{\partial J}{\partial\xi_{0,0}}\right)^{-2} - U_{\mathrm{B}} \right\} \rho d^4x \tag{2.19}$$

$$\rho \equiv \rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,0}}$$

It follows from (2.12) that

$$\rho U_{\mathrm{B}} = \rho U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right) = \frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}\rho)^2}{\rho} - \frac{\hbar^2}{4m}\boldsymbol{\nabla}^2\rho \tag{2.20}$$

The last term of (2.20) has a form of divergence, and it does not contribute to dynamic equations. This term may be omitted.

If the relation

$$\frac{\partial J}{\partial \xi_{0,0}} \neq 0 \tag{2.21}$$

takes place, the variational problems (2.15) and (2.19) are equivalent. On the contrary, if the relation (2.21) is violated we cannot be sure, that they are equivalent.

Now we introduce designation $j = \{j^0, \mathbf{j}\} = \{\rho, \mathbf{j}\} = \left\{j^k\right\}$, $k = 0, 1, 2, 3$

$$j^k = \rho_0(\boldsymbol{\xi}) \frac{\partial\left(x^k, \xi_1, \xi_2, \xi_3\right)}{\partial\left(x^0, x^1, x^2, x^3\right)} = \rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,k}}, \qquad k = 0, 1, 2, 3 \tag{2.22}$$

and add designation (2.22) to the action (2.19) by means the Lagrangian multipliers p_k, $k = 0, 1, 2, 3$. We obtain

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\xi, j, p] = \int_{V_x} \left\{ m\frac{\mathbf{j}^2}{2\rho} - \rho U_{\mathrm{B}} - p_k\left(j^k - \rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial \xi_{0,k}}\right)\right\} d^4x, \tag{2.23}$$

$$U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right), \qquad \rho \equiv j^0$$

Note that the action (2.19) and the action (2.23) describe the same variational problem. The action (2.23) is interesting in the sense, that the Lagrangian coordinates $\xi = \{\xi_0, \boldsymbol{\xi}\}$ are concentrated in the last term of the action. The Lagrangian coordinates $\xi = \{\xi_0, \boldsymbol{\xi}\}$ are defined to within the transformation

$$\xi_0 = f_0\left(\tilde{\xi}_0\right), \qquad \xi_\alpha = f_\alpha\left(\tilde{\boldsymbol{\xi}}\right), \qquad \alpha = 1, 2, 3 \tag{2.24}$$

where f_k, $k = 0, 1, 2, 3$ are arbitrary functions of their arguments. The variable ξ_0 is fictitious, and variation with respect to ξ_0 does not give an independent dynamic equation.

Variation of the action (2.23) with respect to ξ_l, $l = 0, 1, 2, 3$ leads to the dynamic equations

$$\frac{\delta \mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}}{\delta \xi_l} = -\partial_s\left(\rho_0(\boldsymbol{\xi}) p_k \frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{l,s}}\right) + p_k\frac{\partial \rho_0}{\partial \xi_l}(\boldsymbol{\xi})\frac{\partial J}{\partial \xi_{0,k}} = 0, \qquad l = 0, 1, 2, 3 \tag{2.25}$$

Using identities

$$\frac{\partial J}{\partial \xi_{i,l}}\xi_{k,l} \equiv J\delta_k^i, \qquad \partial_l \frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{i,l}} \equiv 0 \tag{2.26}$$

$$\frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{l,s}} \equiv J^{-1}\left(\frac{\partial J}{\partial \xi_{0,k}}\frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}}\frac{\partial J}{\partial \xi_{l,k}}\right) \tag{2.27}$$

we obtain from (2.25) by means of identities (2.27), (2.26)

$$-\frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{l,s}}\rho_0 \partial_s p_k - \frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{l,s}}\frac{\partial \rho_0}{\partial \xi_j}\xi_{j,s}p_k + p_k \frac{\partial \rho_0}{\partial \xi_l}\frac{\partial J}{\partial \xi_{0,k}} = 0, \qquad l = 0,1,2,3$$

$$-J^{-1}\left(\frac{\partial J}{\partial \xi_{0,k}}\frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}}\frac{\partial J}{\partial \xi_{l,k}}\right)\rho_0(\boldsymbol{\xi})\,\partial_s p_k - \left(\frac{\partial J}{\partial \xi_{0,k}}\delta_j^l - \delta_j^0 \frac{\partial J}{\partial \xi_{l,k}}\right)\frac{\partial \rho_0(\boldsymbol{\xi})}{\partial \xi_j}p_k$$
$$+p_k \frac{\partial \rho_0(\boldsymbol{\xi})}{\partial \xi_l}\frac{\partial J}{\partial \xi_{0,k}} = 0, \qquad l = 0,1,2,3 \tag{2.28}$$

Simplifying (2.28) by means of the first identity (2.26), we obtain

$$J^{-1}\left(\frac{\partial J}{\partial \xi_{0,k}}\frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}}\frac{\partial J}{\partial \xi_{l,k}}\right)\rho_0 \partial_s p_k = 0 \tag{2.29}$$

Convoluting (2.29) with $\xi_{l,i}$ and using the first identity (2.26) and designations (2.22), we obtain

$$j^k \partial_i p_k - j^k \partial_k p_i = 0, \qquad i = 0,1,2,3 \tag{2.30}$$

Variation of (2.23) with respect to j^β gives

$$\delta j^\beta: \quad p_\beta = m\frac{j^\beta}{\rho}, \qquad \beta = 1,2,3 \tag{2.31}$$

Varying (2.23) with respect to $j^0 = \rho$, using designations

$$\rho_\gamma \equiv \partial_\gamma \rho, \qquad \rho_{\alpha\beta} \equiv \partial_\alpha \partial_\beta \rho$$

and taking into account relation (2.12) for $U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right)$, we obtain

$$\begin{aligned} \delta j^0 \quad : \quad p_0 &= -\frac{m}{2\rho^2}j^\alpha j^\alpha - \frac{\partial}{\partial \rho}(\rho U_{\mathrm{B}}) + \partial_\gamma \frac{\partial}{\partial \rho_\gamma}(\rho U_{\mathrm{B}}) - \partial_\alpha \partial_\beta \frac{\partial}{\partial \rho_{\alpha\beta}}(\rho U_{\mathrm{B}}) \\ &= -\frac{m}{2\rho^2}j^\alpha j^\alpha - U_{\mathrm{B}} \end{aligned} \tag{2.32}$$

We note the remarkable property of the Bohm potential $U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right)$, defined by the relation (2.12). The quantity p_0 is expressed via $U_{\mathrm{B}} = U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right)$ in such a way, as if $U\left(\rho, \boldsymbol{\nabla}\rho, \boldsymbol{\nabla}^2\rho\right)$ does not depend on ρ and its derivatives.

Eliminating p_k from the equations (2.30) by means of relations (2.31), (2.32) and setting $\mathbf{v} = \mathbf{j}/\rho$, we obtain dynamic equations in the Eulerian form (2.3).

There is another possibility. The dynamic equations (2.29) may be considered to be linear partial differential equations with respect to variables p_k. They can be solved in the form

$$p_k = b_0\left(\partial_k \varphi + g^\alpha(\boldsymbol{\xi})\,\partial_k \xi_\alpha\right), \qquad k = 0,1,2,3 \tag{2.33}$$

where $g^{\alpha}(\boldsymbol{\xi})$, $\alpha = 1,2,3$ are arbitrary functions of the argument $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$, $b_0 \neq 0$ is an arbitrary real constant, and φ is the variable ξ_0, which ceases to be fictitious. Note that the constant b_0 may be eliminated, including it in the functions $\mathbf{g} = \{g^1, g^2, g^3\}$ and in the variable φ.

One can test by the direct substitution that the relation (2.33) is the general solution of linear equations (2.29). Substituting (2.33) in (2.29) and taking into account antisymmetry of the bracket in (2.29) with respect to transposition of indices k and s, we obtain

$$J^{-1}\rho_0(\boldsymbol{\xi})\left(\frac{\partial J}{\partial \xi_{0,k}}\frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}}\frac{\partial J}{\partial \xi_{l,k}}\right)\frac{\partial g^{\alpha}(\boldsymbol{\xi})}{\partial \xi_{\mu}}\xi_{\mu,s}\xi_{\alpha,k} = 0 \tag{2.34}$$

The relation (2.34) is a valid equality, as it follows from the first identity (2.26).

Let us substitute (2.33) in the action (2.23). Taking into account the first identity (2.26) and omitting the term

$$\rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial \xi_{0,k}}\partial_k\varphi = \rho_0(\boldsymbol{\xi})\frac{\partial(\varphi, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)}$$

which does not contribute to the dynamic equations, we obtain

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\varphi, \boldsymbol{\xi}, j] = \int \left\{\frac{m}{2}\frac{j^{\alpha}j^{\alpha}}{j^0} - U_{\mathrm{B}}\rho - j^k b_0\left(\partial_k\varphi + g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_{\alpha}\right)\right\} d^4x, \qquad j^0 \equiv \rho \tag{2.35}$$

Variation of (2.35) with respect to $j^0 \equiv \rho$ gives

$$-\frac{m\mathbf{j}^2}{2\rho^2} - U_{\mathrm{B}} - b_0\left(\partial_0\varphi + g^{\alpha}(\boldsymbol{\xi})\partial_0\xi_{\alpha}\right) = 0, \qquad U_{\mathrm{B}} = \frac{\hbar^2}{8m}\left(\frac{(\boldsymbol{\nabla}\rho)^2}{\rho^2} - 2\frac{\boldsymbol{\nabla}^2\rho}{\rho}\right) \tag{2.36}$$

Variation of (2.35) with respect to j^{μ} gives

$$m\frac{j^{\mu}}{\rho} = b_0\left(\partial_{\mu}\varphi + g^{\alpha}(\boldsymbol{\xi})\partial_{\mu}\xi_{\alpha}\right) \tag{2.37}$$

Variation of (2.35) with respect to φ gives

$$\partial_k j^k = 0 \tag{2.38}$$

Finally, varying (2.35) with respect to ξ_{μ} and taking into account (2.38), we obtain

$$b_0 j^k \Omega^{\alpha\mu}(\boldsymbol{\xi})\partial_k\xi_{\alpha} = 0, \qquad \Omega^{\alpha\mu}(\boldsymbol{\xi}) = \left(\frac{\partial g^{\alpha}(\boldsymbol{\xi})}{\partial \xi_{\mu}} - \frac{\partial g^{\mu}(\boldsymbol{\xi})}{\partial \xi_{\alpha}}\right) \tag{2.39}$$

If

$$\det\|\Omega^{\alpha\mu}\| \neq 0 \tag{2.40}$$

then taking into account that the velocity $\mathbf{v} = \mathbf{j}/j^0$, one obtains from (2.39), so called Lin constraint [17]

$$\partial_0\boldsymbol{\xi} + (\mathbf{v}\boldsymbol{\nabla})\boldsymbol{\xi} = 0 \tag{2.41}$$

which means that the variables $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ are constant along the mean world lines of particles. In other words, the variables $\boldsymbol{\xi}$ are Lagrangian coordinates, which label mean world lines of particles.

However, the constraint (2.40) does not take place always. In particular, $\Omega^{\alpha\beta} \equiv 0$ in the case of irrotational flow. Besides, the quantity $\Omega^{\alpha\beta}$ is antisymmetric, as it follows from the second relation (2.39), and

$$\det \left|\left|\Omega^{\alpha\beta}\right|\right| = \left| \begin{array}{ccc} 0 & \Omega^{12} & \Omega^{13} \\ -\Omega^{12} & 0 & \Omega^{23} \\ -\Omega^{13} & -\Omega^{23} & 0 \end{array} \right| \equiv 0 \tag{2.42}$$

Note that identity (2.42) is a property of the three-dimensional space. In the two-dimensional space $\det \left|\left|\Omega^{\alpha\beta}\right|\right| = \left(\Omega^{12}\right)^2$. In the case of four-dimensional space we have

$$\det \left|\left|\Omega^{\alpha\beta}\right|\right| = \left(\Omega^{12}\Omega^{34} - \Omega^{13}\Omega^{24} + \Omega^{14}\Omega^{23}\right)^2$$

It seems rather strange and unexpected, that the Lin constraint (2.41) is not a corollary of the dynamic equation (2.39), although the Lin constraint (2.41) is compatible with the dynamic equation (2.39). In the case of nonrotational flow the Euler hydrodynamic equations for perfect fluid can be obtained from the variational principle [12]. In the case of a rotational flow of the same fluid the Euler hydrodynamic equations can be deduced from the variational principle, only when the Lin constraints are introduced in the action functional as side conditions, and the variables $\boldsymbol{\xi}$ are considered as dynamic variables [17]. Does it mean, that the Lagrangian coordinates $\boldsymbol{\xi}$ are inadequate dynamical variables? Maybe. It is not clear now.

From equations (2.36) - (2.41) one obtains five equations

$$-\frac{\left(\boldsymbol{\nabla}\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\alpha}\right)^2}{2m} - U_{\mathrm{B}} - \left(\partial_0\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\partial_0\xi_{\alpha}\right) = 0, \tag{2.43}$$

$$\partial_0\boldsymbol{\xi} + \left(\mathbf{v}\boldsymbol{\nabla}\right)\boldsymbol{\xi} = 0 \tag{2.44}$$

$$\partial_0\rho + \boldsymbol{\nabla}\left(\rho\frac{\left(\boldsymbol{\nabla}\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\alpha}\right)}{m}\right) \tag{2.45}$$

for five dynamic variables $\rho, \varphi, \boldsymbol{\xi}$. Indefinite functions $\mathbf{g}\left(\boldsymbol{\xi}\right) = \{g^1\left(\boldsymbol{\xi}\right), g^2\left(\boldsymbol{\xi}\right), g^3\left(\boldsymbol{\xi}\right)\}$ are determined from initial conditions for velocity $\mathbf{v} = \mathbf{j}/\rho$. The constant b_0 is included in the indefinite functions $\varphi, \mathbf{g}\left(\boldsymbol{\xi}\right)$ The velocity $\mathbf{v}$ is expressed via dynamic variables $\rho, \varphi, \boldsymbol{\xi}$ by means of the relation

$$\mathbf{v} = \frac{\mathbf{j}}{\rho} = \frac{\left(\boldsymbol{\nabla}\varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\boldsymbol{\nabla}\xi_{\alpha}\right)}{m} \tag{2.46}$$

3 Description in terms of wave function

Clebsch potentials $\boldsymbol{\xi}$, φ and the density ρ can be used for formation of a complex wave function ψ. By means of a change of variables the action (2.35) can be transformed to a description in terms of a wave function [1]. Let us introduce the k-component complex function $\psi=\{\psi_\alpha\}$, $\alpha=1,2,...k$, defining it by the relations

$$\psi_\alpha=\sqrt{\rho}e^{i\varphi}u_\alpha(\boldsymbol{\xi}),\qquad \psi_\alpha^*=\sqrt{\rho}e^{-i\varphi}u_\alpha^*(\boldsymbol{\xi}),\qquad \alpha=1,2,...k \tag{3.1}$$

$$\psi^*\psi\equiv\sum_{\alpha=1}^{k}\psi_\alpha^*\psi_\alpha \tag{3.2}$$

where (*) means the complex conjugate, $u_\alpha(\boldsymbol{\xi})$, $\alpha=1,2,...k$ are functions of only variables $\boldsymbol{\xi}$. They satisfy the relations

$$-\frac{i}{2}\sum_{\alpha=1}^{k}(u_\alpha^*\frac{\partial u_\alpha}{\partial\xi_\beta}-\frac{\partial u_\alpha^*}{\partial\xi_\beta}u_\alpha)=g^\beta(\boldsymbol{\xi}),\qquad \beta=1,2,...k\qquad \sum_{\alpha=1}^{k}u_\alpha^*u_\alpha=1 \tag{3.3}$$

where k is such a natural number that equations (3.3) admit a solution. In general, k depends on the form of the arbitrary functions $\mathbf{g}=\{g^\beta(\boldsymbol{\xi})\}$, $\beta=1,2,3$.

It is easy to verify, that

$$\rho=\psi^*\psi,\qquad j^\mu=-\frac{ib_0}{2m}(\psi^*\partial_\mu\psi-\partial_\mu\psi^*\cdot\psi),\qquad \mu=1,2,3 \tag{3.4}$$

The variational problem with the action (2.35) appears to be equivalent to the variational problem with the action functional [1]

$$\begin{aligned}\mathcal{A}[\psi,\psi^*] &= \int\left\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\psi)+\frac{b_0^2(\psi^*\boldsymbol{\nabla}\psi-\boldsymbol{\nabla}\psi^*\cdot\psi)^2}{8m\psi^*\psi}\right.\\ &\left.-\frac{\hbar^2}{8m}\frac{\left(\boldsymbol{\nabla}\left(\psi^*\psi\right)\right)^2}{\psi^*\psi}\right\}\mathrm{d}^4x\end{aligned} \tag{3.5}$$

where $\boldsymbol{\nabla}=\{\partial_\alpha\}$, $\alpha=1,2,3$.

Let us consider the case, when the number k of the wave function components is equal to 2. In this case the wave function $\psi=\left\{\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right\}$ has four real components. The number of hydrodynamic variables ρ, $\mathbf{j}$ is also four, and we may hope that the first three equations (3.3) can be solved for any choice of functions $\mathbf{g}$. For the two-component wave function ψ we have the identity

$$(\psi^*\boldsymbol{\nabla}\psi-\boldsymbol{\nabla}\psi^*\cdot\psi)^2\equiv-4\rho\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi+\left(\boldsymbol{\nabla}\rho\right)^2+4\rho^2\sum_{\alpha=1}^{3}\left(\boldsymbol{\nabla}s_\alpha\right)^2 \tag{3.6}$$

where

$$\rho=\psi^*\psi,\qquad s_\alpha=\frac{\psi^*\sigma_\alpha\psi}{\rho},\qquad \alpha=1,2,3 \tag{3.7}$$

σ_α are 2×2 Pauli matrices

$$\sigma_1=\left(\begin{array}{cc}0&1\\1&0\end{array}\right),\qquad \sigma_2=\left(\begin{array}{cc}0&-i\\i&0\end{array}\right),\qquad \sigma_3=\left(\begin{array}{cc}1&0\\0&-1\end{array}\right), \tag{3.8}$$

Substituting (3.6) in (3.5), we obtain

$$\begin{aligned}\mathcal{A}[\psi,\psi^*] &= \int\left\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{b_0^2}{2m}\nabla\psi^*\cdot\nabla\psi+\frac{b_0^2}{8m}\rho\left(\nabla s_\alpha\right)^2\right.\\ &\left.+\frac{b_0^2}{8m}\frac{\left(\nabla\rho\right)^2}{\rho}-\frac{\hbar^2}{8m}\frac{\left(\nabla\rho\right)^2}{\rho}\right\}\mathrm{d}^4x \end{aligned}\tag{3.9}$$

If we choose the arbitrary constant b_0 in the form $b_0=\hbar$, the action (3.9) takes the form

$$\begin{aligned}\mathcal{A}[\psi,\psi^*] &= \int\left\{\frac{i\hbar}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi\right.\\ &\left.+\frac{\hbar^2}{8m}\rho\nabla s_\alpha\nabla s_\alpha\right\}\mathrm{d}^4x\end{aligned}\tag{3.10}$$

In the case, when the wave function ψ is one-component, for instance $\psi=\left\{\begin{smallmatrix}\psi_1\\0\end{smallmatrix}\right\}$, or $\psi_1=a\psi_2$, $a=$const, the quantities $\mathbf{s}=\{s_1,s_2,s_3\}$ are constant ($s_1=0,\quad s_2=0,\quad s_3=1$), the action (3.10) turns into

$$\mathcal{A}[\psi,\psi^*]=\int\left\{\frac{i\hbar}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi\right\}\mathrm{d}^4x\tag{3.11}$$

The dynamic equation, generated by the action (3.11), is the Schrödinger equation

$$i\hbar\partial_0\psi+\frac{\hbar^2}{2m}\nabla^2\psi=0\tag{3.12}$$

This dynamic equation describes the flow of the fluid.

In the general case the dynamic equation, generated by the action (3.10) has the form

$$i\hbar\partial_0\psi+\frac{\hbar^2}{2m}\nabla^2\psi+\frac{\hbar^2}{8m}\nabla^2 s_\alpha\cdot\left(s_\alpha-2\sigma_\alpha\right)\psi-\frac{\hbar^2}{4m}\frac{\nabla\rho}{\rho}\nabla s_\alpha\sigma_\alpha\psi=0\tag{3.13}$$

Deriving dynamic equation (3.13), we have used the identities

$$\mathbf{s}^2\equiv1,\qquad s_\alpha\nabla s_\alpha\equiv0,\qquad \nabla s_\alpha\left(\nabla s_\alpha\right)+s_\alpha\nabla^2 s_\alpha\equiv0$$

Using the change of variables (3.1), (3.3), we did not use the fact, that the solution of equations (2.39) is a solution of the equations (2.41). In the case of description in terms of the wave function ψ we have not the problem, which we have at description in terms of the generalized stream function $\boldsymbol{\xi}$, when there are such solutions of (2.39), which are not solutions of (2.41).

Thus, we have seen that a consecutive application of the relativity theory (relativistic concept of the particle state) admits one to describe the mean motion of a stochastic particle $\mathcal{S}_{\rm st}$ in terms of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$. The statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$ is a dynamic system of the type of continuous medium. Dynamics of the statistical ensemble is described by dynamic equations of the hydrodynamic type. These dynamic equations may be described in terms of the wave function. In the special case of the internal energy of the fluid, describing $\mathcal{E}[\mathcal{S}_{\rm st}]$, has the form $E_{\rm int} = \frac{\hbar^2}{8m^2}\left(\frac{\boldsymbol{\nabla}\boldsymbol{\rho}}{\rho}\right)^2$, the dynamic equation coincides with the Schrödinger equation, if the fluid flow is irrotational. In the general case of vortical flow the dynamic equation in terms of the wave function is not linear. The dynamic equations for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$ have been derived without a reference to the quantum principles (in particular, to the linearity principle). In other types of the particle stochasticity one can obtain dynamic equations for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$ of stochastic particles $\mathcal{S}_{\rm st}$.

4 Multivariance of the space-time geometry

The real space-time geometry is multivariant in the sense that at a point P there are many vectors $\mathbf{PQ}$, $\mathbf{PQ}'$, $\mathbf{PQ}''$,... which are equivalent to the vector $\mathbf{AB}$ at the point A, but vectors $\mathbf{PQ}$, $\mathbf{PQ}'$, $\mathbf{PQ}''$,... are not equivalent between themselves. The property of multivariance is generated by intransitivity of the equivalence relation (1.1), (1.5), (1.6) of two vectors. This definition of the vector equivalence (equality) is given without a reference to the method of the geometry description (coordinate system). It is more correct, than the conventional definition of the vector equivalence, when vectors are equivalent (equal), if their coordinates are equal. In the proper Euclidean geometry both definitions coincide, because the proper Euclidean geometry is single-variant. However, in the geometry of Minkowski (pseudo-Euclidean geometry of index 1) both definitions coincide only for timelike vectors. The geometry of Minkowski is multivariant with respect to spacelike vectors. The spacelike world lines wobble with infinite amplitude. This circumstance is a reason, why a single tachyon (i.e. the particle with spacelike world line) cannot be detected. If one uses the conventional definition of vector equivalence (equality of vector coordinates), the world line of a tachyon is considered as a smooth (not wobbling). In this case experimental impossibility of the tachyon detection is interpreted as a proof of the statement, that tachyons do not exist. In the Riemannian space-time geometry mathematicians were forced to forbid fernparallelism and to introduce the parallel transport of vectors, in order to suppress the natural multivariance of the Riemannian space-time geometry with respect to outlying vectors.

In general, the coordinateless metric approach to the space-time geometry, when the geometry is described completely by the world function is more reasonable, than conventional method of the geometry description, when the space-time geometry description begins from fixation of a dimension and of a coordinate system.

Any generalized geometry $\mathcal{G}$ is a generalization of the proper Euclidean geometry $\mathcal{G}_{\rm E}$. This generalization depends essentially on the presentation of $\mathcal{G}_{\rm E}$. Conventional presentation of $\mathcal{G}_{\rm E}$ (V-presentation) contains several basic concepts: (1) dimension, (2) coordinate system, (3) infinitesimal distance, (4) equivalence of vectors. In $\mathcal{G}_{\rm E}$ all these concepts are coherent, and axioms, connecting these concepts are consistent. At a generalization of $\mathcal{G}_{\mathrm{E}}$ the basic concepts are modified. Corresponding modification of axioms must be such, that the modified axioms were consistent. It is a very difficult problem, because there exists a lot of generalized geometries, and for all of them the axioms are to be consistent.

To solve the problem of $\mathcal{G}_{\mathrm{E}}$ generalization one should use a monistic presentation of $\mathcal{G}_{\mathrm{E}}$, when there is only one basic concept, and all other geometric concepts and quantities are derivative. They are obtained from the unique basic concept. Such a monistic presentation of $\mathcal{G}_{\mathrm{E}}$ is the σ-presentation, when $\mathcal{G}_{\mathrm{E}}$ is described in terms and only in terms of the world function σ. There are connections between the world function σ and derivative geometrical concepts. These connections are conserved at the $\mathcal{G}_{\mathrm{E}}$ generalization, provided these connections can be expressed in terms of the world function and only in terms of the world function. However, the world function σ_{E} of $\mathcal{G}_{\mathrm{E}}$ may have its specific properties, which are not conserved at a generalization of $\mathcal{G}_{\mathrm{E}}$.

At the generalization of the proper Euclidean geometry one obtains a physical geometry $\mathcal{G}$, replacing the world function σ_{E} by the world function σ of the geometry $\mathcal{G}$ in all geometric relations of $\mathcal{G}_{\mathrm{E}}$, which can be expressed in terms of only the Euclidean world function σ_{E}. These relations will referred to as general geometric relations. Expressions (1.1), (1.5), (1.6) are examples of general geometric relations.

Another example of such a relation is definition of linear dependence of n vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$. Vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$ are linear dependent, if and only if the condition

$$F_n\left(\mathcal{P}^n\right) = 0 \tag{4.1}$$

is fulfilled. Here $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ and $F_n\left(\mathcal{P}^n\right)$ is the Gram determinant

$$F_n\left(\mathcal{P}^n\right) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)|| , \quad i, k = 1, 2, ...n \tag{4.2}$$

Scalar product in (4.2) is expressed via the world function by means of (1.5).

5 Specific properties of the proper Euclidean geometry

There are specific properties of $\mathcal{G}_{\mathrm{E}}$, which are not conserved at the replacement of σ_{E} by σ. If $\sigma = \sigma_{\mathrm{E}}$ is the world function of n-dimensional Euclidean space E^n, it satisfies the following relations.

I. Definition of the dimension and introduction of the rectilinear coordinate system:

$$\exists \mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \qquad F_n\left(\mathcal{P}^n\right) \neq 0, \qquad F_k\left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{5.1}$$

where $F_n\left(\mathcal{P}^n\right)$ is the Gram's determinant (4.2), and Ω is the set of points, where the geometry is given. Vectors $\mathbf{P}_0\mathbf{P}_i, \ \ i = 1, 2, ...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0. The covariant metric tensor

$g_{ik}(\mathcal{P}^n)$, $i,k=1,2,...n$ and the contravariant one $g^{ik}(\mathcal{P}^n)$, $i,k=1,2,...n$ in a rectilinear coordinate system K_n are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}(\mathcal{P}^n) g_{lk}(\mathcal{P}^n) = \delta_l^i, \qquad g_{il}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i,l = 1,2,...n \tag{5.2}$$

$$F_n(\mathcal{P}^n) = \det ||g_{ik}(\mathcal{P}^n)|| \neq 0, \qquad i,k = 1,2,...n \tag{5.3}$$

II. Linear structure of the Euclidean space:

$$\sigma_{\mathrm{E}}(P,Q) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}(\mathcal{P}^n)(x_i(P) - x_i(Q))(x_k(P) - x_k(Q)), \qquad \forall P,Q \in \Omega \tag{5.4}$$

where coordinates $x_i(P)$, $i=1,2,...n$ of the point P are covariant coordinates of the vector $\mathbf{P}_0\mathbf{P}$, defined by the relation

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i = 1,2,...n \tag{5.5}$$

III: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues

$$g_k > 0, \qquad k = 1,2,...,n \tag{5.6}$$

IV. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i = 1,2,...n \tag{5.7}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i=1,2,...n$ has always one and only one solution.

Not all conditions I – IV are independent, they determine different properties of $\mathcal{G}_{\mathrm{E}}$. For instance, the condition I determines the dimension n of the Euclidean space E^n. This dimension n is the maximal number of linear independent vectors in $\mathcal{G}_{\mathrm{E}}$. This number is determined by the general geometric expression (4.2) which depends on the form of the world function. If the world function changes, and conditions (5.1) are not fulfilled, one cannot introduce a coordinate system in the conventional form, because the metric dimension n_m of the generalized geometry $\mathcal{G}$ remains to be not determined. For instance, in the discrete geometry $\mathcal{G}_{\mathrm{d}}$, defined by (1.3) one can find five vectors $\mathbf{PP}_0 = (1,0,0,0)$, $\mathbf{PP}_1 = (0,1,0,0)$, $\mathbf{PP}_2 = (0,0,1,0)$, $\mathbf{PP}_3 = (0,0,0,1)$, $\mathbf{PP}_4 = (a,0,0,0)$, $a > 1$, which are linear dependent. For the case of the discrete geometry (1.3) calculation gives

$$F_4(P,P_0,P_1,P_2,P_3) = -1 - 4\lambda_0^2 + \mathcal{O}(\lambda_0^4), \quad \lambda_0^2 \ll 1 \tag{5.8}$$

$$F_5(P,P_0,P_1,P_2,P_3,P_4) = -\lambda_0^2 - a\lambda_0^2(a-1) + \mathcal{O}(\lambda_0^4), \quad \lambda_0^2 \ll 1 \tag{5.9}$$

We see, that the fifth order Gram determinant in the "4-dimensional" discrete geometry does not vanish. However, it vanishes, if $\lambda_0 \to 0$ and the discrete geometry turns to the geometry of Minkowski. It means that the metric dimension n_m in $\mathcal{G}_{\mathrm{d}}$ is more, than the coordinate dimension n_c (the number of coordinates) $n_c < n_m$. The Gram determinant for two "linear dependent" vectors $\mathbf{PP}_0 = (1,0,0,0)$, $\mathbf{PP}_4 = (a,0,0,0)$, $a > 1$

$$F_2\left(P,P_0,P_4\right) = \lambda_0^2\left(1 - a + a^2 + \frac{3}{4}\lambda_0^2\right) \tag{5.10}$$

Vectors $\mathbf{PP}_0$, $\mathbf{PP}_4$ are linear dependent from viewpoint of their coordinate representation. However, from the viewpoint of the metric approach the vectors $\mathbf{PP}_0$, $\mathbf{PP}_4$ are linear independent.

Thus, specific properties (5.1) of the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ are responsible for dimension and coordinate system in $\mathcal{G}_{\mathrm{E}}$. In the Riemannian geometry the properties (5.1) are fulfilled locally

$$\exists\mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega_\varepsilon, \qquad F_n\left(\mathcal{P}^n\right) \neq 0, \qquad F_k\left(\Omega_\varepsilon^{k+1}\right) = 0, \qquad k > n \tag{5.11}$$

where Ω_ε is infinitesimal region of the total point set Ω, which is placed around the point P_0. For finite vectors the Riemannian geometry is multivariant, generally speaking.

In the geometry of Minkowski $\mathcal{G}_{\mathrm{M}}$ the specific conditions of the $\mathcal{G}_{\mathrm{E}}$ are not fulfilled, because the relation (5.6) is violated. Geometry of Minkowski is multivariant with respect to spacelike vectors, although it is single-variant with respect to timelike vectors.

6 Recognition of geometric objects

The most important problem of a geometry is a recognition of the same geometric object in different space-time geometries. Such a problem arises, when a physical body (a particle) travels from the space-time region Ω_1 with geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ to another space-time region Ω_2 with geometry $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$. How is the geometrical object to be described, in order it may be recognized in different space-time regions Ω_1 and Ω_2? In the conventional approach to geometry such a problem is not considered at all.

We consider this problem in the simplest example, when the geometrical object in geometries $\mathcal{G}_1$ and $\mathcal{G}_2$ is a segment $\mathcal{T}_{[P_0P_1]}$ of a straight line between the points P_0 and P_1. In the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ this segment may be described as a set of points R, defined by the relation

$$\mathcal{T}_{[P_0P_1]} = \left\{R|\rho\left(P_0,R\right) + \rho\left(R,P_1\right) - \rho\left(P_0,P_1\right) = 0\right\}, \qquad \rho = \sqrt{2\sigma} \tag{6.1}$$

where $\sigma = \sigma_{\mathrm{E}}$ is the world function of $\mathcal{G}_{\mathrm{E}}$. The segment in $\mathcal{G}_{\mathrm{E}}$ is one-dimensional in the sense, that a section $S\left(P,\mathcal{T}_{[P_0P_1]}\right)$ of the segment $\mathcal{T}_{[P_0P_1]}$ at any point $P \in \mathcal{T}_{[P_0P_1]}$ consists only of this point P.

$$S\left(P,\mathcal{T}_{[P_0P_1]}\right) \equiv \left\{R|\bigwedge_{s=0,1}\rho\left(P_s,P\right) = \rho\left(P_s,R\right)\right\} = \{P\}, \qquad \rho = \sqrt{2\sigma} \tag{6.2}$$

In the geometry of Minkowski $\mathcal{G}_{\rm M}$ a timelike segment $\mathcal{T}_{[P_0P_1]}$, $(\sigma_{\rm M}(P_0,P_1)>0)$ is described also by the relation (6.1), where $\sigma=\sigma_{\rm M}$. It is also one-dimensional. The spacelike segment $\mathcal{T}_{[P_0P_1]}$, $(\sigma_{\rm M}(P_0,P_1)<0)$ is a three-dimensional infinite surface.

Is a segment of a timelike straight line one-dimensional in other space-time geometries $\mathcal{G}$? One believes, that the segment $\mathcal{T}_{[P_0P_1]}$ of a straight line is one-dimensional in the real space-time geometry, because one believes that a straight line is one-dimensional in any space-time geometry. Such a belief imposes restrictions on the distance function ρ of the geometry $\mathcal{G}_{\rm m}$ (metric geometry), where any straight line is one-dimensional.

On the other hand, let us suppose an ellipsoid in $\mathcal{G}_{\rm E}$. It is determined in terms of the distance ρ in the form

$$\mathcal{EL}_{F_1F_2P}=\{R|\rho(F_1,R)+\rho(R,F_2)=\rho(F_1,P)+\rho(P,F_2)\} \tag{6.3}$$

where F_1,F_2 are focuses of the ellipsoid and P is some point on the ellipsoid surface. If the point P coincides with the focus F_2, the ellipsoid degenerates into the segment $\mathcal{T}_{[F_1P]}$ of the straight line.

$$\mathcal{EL}_{F_1PP}=\mathcal{T}_{[F_1P]}=\{R|\rho(F_1,R)+\rho(R,P)=\rho(F_1,P)\} \tag{6.4}$$

The degenerate ellipsoid $[\mathcal{EL}_{F_1F_2P}]_{F_2=P}$ degenerates into one-dimensional segment in $\mathcal{G}_{\rm E}$ In arbitrary geometry $\mathcal{G}$ it may not remain to be a one-dimensional line. It remains to be one-dimensional, if the triangle axiom takes place

$$\rho(F_1,R)+\rho(R,F_2)\geq\rho(F_1,F_2), \quad \forall F_1,F_2,P\in\Omega \tag{6.5}$$

Which of two properties of a straight line segment should be taken for definition of the segment in the space-time geometry? (one-dimensionality or degenerate ellipsoid?) It is clear that the straight line segment should be defined via ellipsoid, because such a definition does not put restrictions on the world function (and on the distance function). Besides, such a definition is produced in terms of the world function.

However the segment of the straight line is a simplest geometrical object of the proper Euclidean geometry. There are other geometrical objects, whose properties are more complicated, than the properties of the segment.

Geometrical object is defined in the $\mathcal{G}_{\rm E}$ in terms of the world function $\sigma_{\rm E}$. Replacing in this definition the world function $\sigma_{\rm E}$ by the the world function σ of the generalized geometry $\mathcal{G}$, one obtains the definition of this geometric object in $\mathcal{G}$.

Definition 1: A geometrical object $g_{\mathcal{P}_n,\sigma}$ of the geometry $\mathcal{G}=\{\sigma,\Omega\}$ is a subset $g_{\mathcal{P}_n,\sigma}\subset\Omega$ of the point set Ω. This geometrical object $g_{\mathcal{P}_n,\sigma}$ is a set of roots $R\in\Omega$ of the function $F_{\mathcal{P}_n,\sigma}$

$$g_{\mathcal{P}_n,\sigma}=\{R|F_{\mathcal{P}_n,\sigma}(R)=0\}, \quad F_{\mathcal{P}_n,\sigma}: \quad \Omega\rightarrow\mathbb{R} \tag{6.6}$$

where $F_{\mathcal{P}_n,\sigma}$ depends on the point R via world functions of arguments $\{\mathcal{P}_n,R\}=\{P_0,P_1,...P_n,R\}$

$$F_{\mathcal{P}_n,\sigma}: \quad F_{\mathcal{P}_n,\sigma}(R)=G_{\mathcal{P}_n,\sigma}(u_1,u_2,...u_s), \qquad s=\frac{1}{2}(n+1)(n+2) \tag{6.7}$$

$$u_l = \sigma\left(w_i, w_k\right), \quad i,k = 0,1,...n+1, \quad l = 1,2,...\frac{1}{2}\left(n+1\right)\left(n+2\right) \tag{6.8}$$

$$w_k = P_k \in \Omega, \qquad k = 0,1,...n, \qquad w_{n+1} = R \in \Omega \tag{6.9}$$

Here $\mathcal{P}_n = \{P_0, P_1, ..., P_n\} \subset \Omega$ are $n+1$ points which are parameters, determining the geometrical object $g_{\mathcal{P}_n,\sigma}$

$$g_{\mathcal{P}_n,\sigma} = \left\{R | F_{\mathcal{P}_n,\sigma}\left(R\right) = 0\right\}, \qquad R \in \Omega, \qquad \mathcal{P}_n \in \Omega^{n+1} \tag{6.10}$$

$F_{\mathcal{P}_n,\sigma}\left(R\right) = G_{\mathcal{P}_n,\sigma}\left(u_1, u_2, ...u_s\right)$ is a function of $\frac{1}{2}\left(n+1\right)\left(n+2\right)$ arguments u_k and of $n+1$ parameters $\mathcal{P}_n$. The set $\mathcal{P}_n = \{P_0, P_1, ...P_n\} \in \Omega^{n+1}$ of the geometric object parameters will be referred to as the skeleton of the geometrical object. The subset $g_{\mathcal{P}_n,\sigma} \subset \Omega$ will be referred to as the envelope of the skeleton. The skeleton is an analog of a frame of reference, attached rigidly to a physical body. Tracing the skeleton motion, one can trace the motion of the physical body. When a particle is considered as a geometrical object, its motion in the space-time is described by the skeleton $\mathcal{P}_n$ motion. At such an approach (the rigid body approximation) the shape of the envelope is of no importance.

Remark: An arbitrary subset Ω' of the point set Ω is not a geometrical object, generally speaking. It is supposed, that physical bodies may have only a shape of a geometrical object, because only in this case one can identify identical physical bodies (geometrical objects) in different space-time geometries.

Existence of the same geometrical objects in different space-time regions, having different geometries, brings up the question on equivalence of geometrical objects in different space-time geometries. Such a question did not brought up before, because one does not consider such a situation, when a physical body moves from one space-time region to another space-time region, having another space-time geometry. In general, mathematical technique of the conventional space-time geometry (differential geometry) is not applicable for simultaneous consideration of several different geometries of different space-time regions.

We can perceive the space-time geometry only via motion of physical bodies in the space-time, or via construction of geometrical objects corresponding to these physical bodies. As it follows from the *definition 1* of the geometrical object, the function $G_{\mathcal{P}_n,\sigma}$ as a function of its arguments u_k, $k = 1,2,...n\left(n+1\right)/2$ (of world functions of different points) is the same in all physical geometries. It means, that a geometrical object $\mathcal{O}_1$ in the geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ is obtained from the same geometrical object $\mathcal{O}_2$ in the geometry $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$ by means of the replacement of σ_1 by σ_2 in the definition of this geometrical object.

Definition 2: Geometrical object $g_{P'_n,\sigma'}$ ($\mathcal{P}'_n = \{P'_0, P'_1, ..P'_n\}$) in the geometry $\mathcal{G}' = \{\sigma', \Omega'\}$ and the geometrical object $g_{P_n,\sigma}$ ($\mathcal{P}_n = \{P_0, P_1, ..P_n\}$) in the geometry $\mathcal{G} = \{\sigma, \Omega\}$ are similar geometrical objects, if

$$\sigma'\left(P'_i, P'_k\right) = \sigma\left(P_i, P_k\right), \qquad i,k = 0,1,..n \tag{6.11}$$

and the functions $G'_{\mathcal{P}'_n,\sigma'}$ for $g_{P'_n,\sigma'}$ and $G_{\mathcal{P}_n,\sigma}$ for $g_{P_n,\sigma}$ in the formula (6.7) are the same functions of arguments $u_1, u_2, ...u_s$

$$G'_{\mathcal{P}'_n,\sigma'}\left(u_1, u_2, ...u_s\right) = G_{\mathcal{P}_n,\sigma}\left(u_1, u_2, ...u_s\right) \tag{6.12}$$

In this case

$$u_l \equiv \sigma(P_i, P_k) = u'_l \equiv \sigma'(P'_i, P'_k), \quad i,k = 0,1,...n, \quad l = 1,2,..n(n+1)/2 \quad (6.13)$$

The functions $F'_{\mathcal{P}'_n,\sigma'}$ for $g_{P'_n,\sigma'}$ and $F_{\mathcal{P}_n,\sigma}$ for $g_{P_n,\sigma}$ in the formula (6.7) have the same roots, if the relation (6.12) is fulfilled. As a result one-to-one connection between the geometrical objects $g_{P'_n,\sigma'}$ and $g_{P_n,\sigma}$ arises.

As far as the physical geometry is determined by its geometrical objects construction, a physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ can be obtained from some known standard geometry $\mathcal{G}_{\rm st} = \{\sigma_{\rm st}, \Omega\}$ by means of a deformation of the standard geometry $\mathcal{G}_{\rm st}$. Deformation of the standard geometry $\mathcal{G}_{\rm st}$ is realized by the replacement of $\sigma_{\rm st}$ by σ in all definitions of the geometrical objects in the standard geometry $\mathcal{G}_{\rm st}$. The proper Euclidean geometry $\mathcal{G}_{\rm E}$ is an axiomatizable geometry. It has been constructed by means of the Euclidean method as a logical construction. Simultaneously the proper Euclidean geometry is a physical geometry. It may be used as a standard geometry $\mathcal{G}_{\rm st}$. Construction of a physical geometry as a deformation of the proper Euclidean geometry will be referred to as the deformation principle [18, 19, 20]. The most physical geometries are nonaxiomatizable geometries. They can be constructed only by means of the deformation principle.

7 Multivariance of the space-time geometry

Multivariance is an immanent property of a space-time geometry [21]. Even the geometry of Minkowski is multivariant (with respect to spacelike vectors). The real space-time geometry is multivariant also with respect to timelike vectors, and this multivariance is a reason of quantum effects. Mathematical formalism of differential geometry is used for description of the space-time geometry. This formalism cannot be adequate at the description of the space-time geometry, because it is incompatible with the concept of multivariance of a geometry.

Formally one can define the operation of summation of g-vectors in $\mathcal{G}_{\rm d}$, but it will be ambiguous. Indeed, the sum **AC** of two g-vectors **AB** and **BC,** when the end of one g-vector is the origin of other one, is defined as follows

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC} \tag{7.1}$$

The sum $\mathbf{AD}_1$ of two arbitrary g-vectors **AB** and **CD** at the point A is defined as follows

$$\mathbf{AB} + \mathbf{CD} = \mathbf{AB} + \mathbf{BD}_1 = \mathbf{AD}_1, \quad (\mathbf{CD}\mathrm{eqv}\mathbf{BD}_1) \tag{7.2}$$

The g-vector $\mathbf{AD}_1$ is defined by relation (7.2) ambiguously, because the g-vector $\mathbf{BD}_1$ is determined ambiguously by the equivalence relation ($\mathbf{CD}\mathrm{eqv}\mathbf{BD}_1$).

The g-vector $\mathbf{AC} = a\mathbf{AB}$, which is a result of multiplication of g-vector **AB** by a real number a is defined by the relations

$$a\mathbf{AB} = \mathbf{AC}, \quad |\mathbf{AC}| = a\,|\mathbf{AB}|\,, \quad (\mathbf{AB}.\mathbf{AC}) = a\,|\mathbf{AB}|^2 \tag{7.3}$$

Result of multiplication is ambiguous, because, generally speaking, the system of two last equations (7.3) has no unique solution in $\mathcal{G}_{\mathrm{d}}$.

Thus, the mathematical formalism of differential geometry cannot be used in a multivariant geometry. As a result the most scientists do not acknowledge the physical geometry and the metric approach to space-time geometry. Contemporary scientists deal with timelike part of the geometry of Minkowski, which is single-variant and with timelike part of the Riemannian geometry, which is single-variant for infinitesimal vectors. They ignore spacelike vectors of the geometry of Minkowski, and ignore existence of tachyons. As a result they have problems with the dark matter. They consider the Riemannian geometry of space-time as the only possible space-time geometry and ignore other physical geometries of the space-time. However, a use of the metric approach to space-time geometry in general relativity admits one to obtain dynamic equations for the world function directly (not for metric tensor) and to construct the extended general relativity (EGR), which is not restricted by the condition of Riemanianess [22]. In EGR the world function is single-valued, whereas in general relativity the world function is many-valued, generally speaking. Besides, in EGR the black holes do not exist [23], because of the induced antigravitation [24].

8 Perception of multivariance

Intransitive equivalence relations meet objections ("the equivalence relation are to be transitive by definition"). Such an objection arises, because all the time scientists dealt only with the Euclidean method of the geometry construction, which is a logical construction, and where the equivalence relation cannot be intransitive. At the metric approach to geometry a physical geometry is constructed as a deformation of the already constructed Euclidean geometry. The geometry deformation (replacement of σ_{E} by σ) is not a logical procedure. The physical geometry is not a logical construction and it is a nonaxiomatizable geometry, which cannot be constructed by the usual Euclidean method. Many scientists are apt to think, that there are no nonaxiomatizable multivariant geometries. Besides, the metric approach to geometry needs a construction of a new mathematical formalism.

This disregard of multivariance is a manifestation of conceptual problems, connected with transition from description of deterministic motion to description of stochastic motion. The concept of multivariance is connected with mathematical formalism, describing the stochastic particle motion. At first the negative relation of scientific community to stochastic motion appeared in the relation to the Boltzmann's works, who suggested to explain deterministic motion of a continuous medium by stochastic motion of this medium molecules. As far as the stochastic motion and the multivariance of the space-time geometry are connected closely, the disregard of the Boltzmann kinetic equations was a disregard of the multivariance as a reason of the stochastic motion. As far as formalism of the stochastic motion description was practically absent, the mathematical community was apt to ignore stochastic motion and its foundation by the multivariant space-time geometry.

The reason of such a disregard may be illustrated in the example of transition from the Aristotelian mechanics to the Newtonian one. The Aristotelian mechanics is essentially a statics, which investigates condition of a body equilibrium under action of different forces. The Aristotelian mechanics did not contain such concepts as acceleration and inertia. It did not consider a violation of the equilibrium. A motion of a body was considered as a travel of the body with balanced forces acting on the body. A transition to Newtonian mechanics means an introduction of new concepts, such as acceleration and inertia. According to definition of Lee Smolin [25] the Newtonian mechanics is a uniting of the rest and of the motion. Introduction of such concepts as inertia and acceleration needed such a mathematical formalism as infinitesimal calculus. Galilei introduced the concept of inertia, which admits one to explain compatibility of the Earth rotation around its axis with experimental data. However, the corresponding mathematical formalism was absent, and the works of Galilei were not acknowledged. It is very difficult to acknowledge a new concept, which changes essentially the existing theory, especially, if the corresponding mathematical formalism has not been formulated.

Concept of inertia has been acknowledged only after works of I. Newton, who had introduced this concept in the first law of mechanics. Although the first law of mechanics is a special case of the second law, the concept of inertia has been introduced as the first law, in order to stress an importance of the concept of inertia, which was not acknowledged by most of scientists in that time.

Now we have a transition from the deterministic particle motion to the stochastic particle motion. This conceptual transition needs a new concept (multivariance) and a new mathematical formalism. In the new formalism of the skeleton conception the number of dynamic equations differs, generally speaking, from the number of dynamic variables. The particle state, described by the skeleton $\mathcal{P}_n$, contains $4n$ dynamic variables, which satisfy $n(n+1)$ dynamic equations. If $4n > n(n+1)$, for instance, for $n = 1, 2$, a solution of dynamic equations is not unique. It is a reason of the world chain wobbling. The concept of multivariance and a new formalism corresponding to uniting of the deterministic particle dynamics with the stochastic particle dynamics is not acknowledged by the scientific community. Apparently, it is a natural thing, that the scientific community did not acknowledge a new concept, which is introduced on the basis of logical consideration of basic physical principles, but not on the basis of new hypotheses, extracted from experimental data.

9 Applications of skeleton conception in microcosm

Thus, correcting mistakes in basic physical and geometric principles, one succeeded to create the skeleton conception of elementary particles (SCEP), which admits one to investigate arrangement of elementary particles, but not only systematize them. It is a new conception, which can be classified as a uniting of a deterministic particle motion and a stochastic one. Application of SCEP to Dirac equation shows, that the Dirac particle (fermion) has a helical world line with timelike axis. Rotation of a particle in its motion along the helix explains freely its spin and magnetic moment. Although this result is obtained at investigation of the Dirac equation, it can be obtained only in the framework of SCEP, when the Dirac equation is considered as a dynamic equation, describing evolution of the statistical ensemble [26, 27, 28, 29, 30]. World line of a free particle can have a shape of a helix, provided skeleton of the particle consists of three points [31, 32]. Reasonable restriction, that the world function is single-valued, leads to the corollary that the electric charge of any elementary particle is not more, than the elementary charge [33]. This result is known from experiment, but it was not explained by the existing theory of elementary particles.

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