Non-Riemannian model of the space-time responsible for quantum effects

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Abstract

A class of homogeneous isotropic space-time models including pseudo-Euclidean space as a special case is considered. Such a model is chosen, where the particle motion is described in the most adequate way. It means that the world tubes of all free particles (both classical and quantal) are characteristic geometrical structures of the space-time. The world function σ of this model has the form $\sigma = \sigma_{\rm E} + \sigma_0$, $\sigma_{\rm E} > \sigma_0$, where $\sigma_{\rm E}$ is the world function of the pseudo-Euclidean space, σ_0 is a constant responsible for quantum effects. It is proportional to Planck's constant $\hbar$.

1 Introduction

When the gravitation is neglected, then the four-dimensional pseudo-Euclidean space is used as a model of the space-time. It can be motivated as follows. First, some of characteristic geometric structures of the pseudo-Euclidean space (straight lines) occur to be real physical objects. For instance, world lines of free macroscopic particles are timelike straights, i.e., characteristic geometrical structures of the pseudo-Euclidean space. Second, we have no homogeneous models of the space-time other than pseudo-Euclidean space and the constant curvature space.

But world lines of free microparticles (electrons, positrons, etc.) are stochastic. They are not characteristic structures of the pseudo-Euclidean space. A description of the microparticle motion needs use of the quantum mechanics principles.

Is it possible such an isotropic homogeneous model of the space-time, that its characteristic geometric structures (analog of timelike straights) would describe the motion of both macro- and microparticles? If so, then the quantum features of the particle motion would be found in the space-time model in itself, the quantum constant $\hbar$ being one of the parameters of this model. Such a model would be attractive, as far as it would not need a use of quantum principles.

Construction of such a model needs rather wide class of isotropic homogeneous space-time models, what, in turn, needs to be beyond of scope of the Riemannian geometry. Recently [1] such a rather strong generalization of the Riemannian geometry appeared. This generalization permits us to construct a rather wide class of isotropic homogeneous models and to realize the above program. It is the σ spaces defined as follows.

Definition 1.1: The σ space $V = (\Omega, \sigma)$ is a set Ω of points P with a real function σ of any pair of points $P, Q \in \Omega$.The function σ has the properties

$$\sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P), \qquad P,Q \in \Omega \tag{1.1}$$

It is called the world function or merely σ function.

Interval

$$S(P,Q) = \sqrt{2\sigma(P,Q)} = \begin{cases} \left|\sqrt{2\sigma(P,Q)}\right|, & \sigma(P,Q) \geq 0 \\ i\left|\sqrt{2\sigma(P,Q)}\right|, & \sigma(P,Q) < 0 \end{cases} \tag{1.2}$$

between points P, Q is called timelike, spacelike or null, if correspondently the following conditions $\sigma(P,Q) > 0$, $\sigma(P,Q) < 0$, or $\sigma(P,Q) = 0 \wedge P \neq Q$ are fulfilled.

As a rule a set $\mathcal{P}^n = \{P_0, P_1, ..., P_n\} \subset \Omega$ of different points $P_0, P_1, ..., P_n$, determines a geometrical object $\mathcal{T}(\mathcal{P}^n)$ of the space V. This object is called the tube of nth order. It is an analog of n-dimensional plane determined by $n+1$ different points in the Euclidean space.

Definition of the σ space realizes the simple physical idea that the space-time properties are described completely by giving interval for every pair of points. It agrees entirely with the conventional definition of geometry (for instance, Riemannian geometry). But at the conventional approach the relations of type (1.1) are given on a manifold $\mathbb{M}$ of a definite dimension (not on an arbitrary set Ω). On the manifold $\mathbb{M}$ such concepts as continuity, curve, surface, and coordinate system are defined. In other words, at the conventional approach both affine relations realized in the concept of manifold and metric relations of type (1.1) are introduced together. They have to be agreed. Only in this case the Riemannian geometry arises.

First, the world function was introduced by Ruse [2, 3] and Synge [4] for describing the Riemannian space. Now it is used mainly in quantum gravitation [5, 6]. In all cases it was used as some derivative structure, i.e., a manifold $\mathbb{M}$ and a coordinate system K on $\mathbb{M}$ were defined first. Then a metric tensor is determined in this coordinate system. Thereafter, the world function is defined as a half of square of interval measured along geodesic.

In the σ space the affine properties introduced by means of a manifold happen to be derivatives of metric properties [1] described by the world function. In particular, if σ space satisfies some conditions (formulated in terms of σ function), then the

σ space is a Riemannian space. It means that the set Ω is a manifold, and one can determine its dimension in terms of σ function, construct a coordinate system, geodesics, etc. For the Riemannian space the tubes $\mathcal{T}(\mathcal{P}^n)$ of nth order are n-dimensional geodesic surfaces.

If the σ space does not satisfy these conditions, it is not a Riemannian space. In this case some generalization of the Riemannian geometry arises. It can be used as a model of the space-time.

Presentation of the σ space properties and the proof of the σ function self-sufficiency can be found in [7]. Here, we shall demonstrate only how to construct a timelike tube $\mathcal{T}_{P_0P_1}$ of the first order in an arbitrary σ space. Such a tube is an analog of the timelike straight line in the pseudo-Euclidean space. Further, the timelike tube $\mathcal{T}_{P_0P_1}$, $\sigma(P_0,P_1)$ will be considered as a world tube of a real free particle, i.e., as a geometrical object describing the real particle behavior.

Definition 1.2: Vector $\mathbf{P}_0\mathbf{P}$ is an ordered set $\{P_0,P\}$ of two points $P_0,P\in\Omega$. $P_0\in\Omega$ is the origin and $P\in\Omega$ is the end of the vector.

Definition 1. 3: Scalar product $(\mathbf{P}_0\mathbf{P}\cdot\mathbf{P}_0\mathbf{Q})$ of two vectors $\mathbf{P}_0\mathbf{P},$ $\mathbf{P}_0\mathbf{Q}$ is the number

$$(\mathbf{P}_0\mathbf{P}\cdot\mathbf{P}_0\mathbf{Q})=\sigma(P_0,P)+\sigma(P_0,Q)-\sigma(P,Q) \tag{1.3}$$

If $P=Q$, then the quantity $|\mathbf{P}_0\mathbf{P}|$ defined by the relation

$$|\mathbf{P}_0\mathbf{P}|^2=(\mathbf{P}_0\mathbf{P}\cdot\mathbf{P}_0\mathbf{P})=2\sigma(P_0,P) \tag{1.4}$$

is referred to as the length of the vector $\mathbf{P}_0\mathbf{P}$. The length of a timelike vector $(\sigma(P_0,P)>0)$ is positive and that of a spacelike vector $(\sigma(P_0,P)<0)$ is imaginary.

Definition 1.4: The vector $\mathbf{P}_0\mathbf{P}$ is parallel to timelike vector $\mathbf{P}_0\mathbf{P}_1$, $(\mathbf{P}_0\mathbf{P}\upuparrows\mathbf{P}_0\mathbf{P})$, if

$$(\mathbf{P}_0\mathbf{P}\cdot\mathbf{P}_0\mathbf{P}_1)=|\mathbf{P}_0\mathbf{P}|\cdot|\mathbf{P}_0\mathbf{P}_1| \tag{1.5}$$

Definition 1.5: Vector $\mathbf{P}_0\mathbf{P}$ is antiparallel to the timelike vector $\mathbf{P}_0\mathbf{P}_1$ $(\mathbf{P}_0\mathbf{P}\uparrow\downarrow\mathbf{P}_0\mathbf{P}_1)$, if

$$(\mathbf{P}_0\mathbf{P}\cdot\mathbf{P}_0\mathbf{P}_1)=-|\mathbf{P}_0\mathbf{P}|\cdot|\mathbf{P}_0\mathbf{P}_1| \tag{1.6}$$

Definition 1.6: Timelike vectors $\mathbf{P}_0\mathbf{P}_1$, and $\mathbf{P}_0\mathbf{P}$ are collinear $(\mathbf{P}_0\mathbf{P}\parallel\mathbf{P}_0\mathbf{P}_1)$, if they are either parallel or antiparallel, that means

$$F_2(P_0,P_1,P_2)\equiv\left|\begin{array}{cc}(\mathbf{P}_0\mathbf{P}_1\cdot\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_1\cdot\mathbf{P}_0\mathbf{P})\\ (\mathbf{P}_0\mathbf{P}\cdot\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}\cdot\mathbf{P}_0\mathbf{P})\end{array}\right|=0 \tag{1.7}$$

The timelike vector $\mathbf{P}_0\mathbf{P}_1$ determines the following sets of points P

$$\mathcal{T}_{P_0P_1}=\{P|\mathbf{P}_0\mathbf{P}\parallel\mathbf{P}_0\mathbf{P}_1\} \tag{1.8}$$

$$\mathcal{T}_{[P_0P_1}=\{P|\mathbf{P}_0\mathbf{P}\upuparrows\mathbf{P}_0\mathbf{P}_1\},\qquad \mathcal{T}_{P_0]P_1}=\{P|\mathbf{P}_0\mathbf{P}\uparrow\downarrow\mathbf{P}_0\mathbf{P}_1\} \tag{1.9}$$

$$\mathcal{T}_{[P_0P_1]}=\{P|S(P_0,P)+S(P,P_1)=S(P_0,P_1)\} \tag{1.10}$$

Here $\mathcal{T}_{P_0P_1}$ is the tube of the first order. $\mathcal{T}_{[P_0P_1}$ and $\mathcal{T}_{P_0]P_1}$ are two tube rays, and $\mathcal{T}_{[P_1P_2]}$ is the tube segment consisting of points that lie on $\mathcal{T}_{P_0P_1}$, between the points P_0,P_1.

2 The tubular model of the space-time

Now let us construct the timelike tube in some special σ space V_4, which is defined as follows. Let $E_4 = (\mathbb{M}_4, \sigma_{\mathrm{E}})$ be four-dimensional pseudo-Euclidean space of index 1. Hence, the set $\mathbb{M}_4$ is a manifold, and σ_{E} is the world function of the pseudo-Euclidean space. In the Galilean coordinate system it can be presented in the form

$$\sigma_{\mathrm{E}}(P,P') = \sigma_{\mathrm{E}}(x,x') = \frac{1}{2} g_{ik}\left(x^i - x'^i\right)\left(x^k - x'^k\right), \qquad S_{\mathrm{E}} = \sqrt{2\sigma_{\mathrm{E}}} \tag{2.1}$$

$$g_{ik} = \mathrm{diag}\left(c^2, -1 - 1, -1\right) \tag{2.2}$$

Now let us define σ space $V_4 = (\mathbb{M}_4, \sigma_{\mathrm{E}})$ on the set $\mathbb{M}_4$ with the world function

$$\sigma = f(\sigma_{\mathrm{E}}) \tag{2.3}$$

where f is some real function (mapping function). We shall use another function g which is connected with the function f. It is defined by the formula

$$S_{\mathrm{E}} = g(S) = (1 - \alpha(S))\, S, \qquad S = \sqrt{2\sigma}, \qquad S_{\mathrm{E}} = \sqrt{2\sigma_{\mathrm{E}}}$$

$$\lim_{S\to 0} \frac{g^2(S)}{S^2} = \frac{1}{2}, \qquad \lim_{S\to+\infty} \frac{g^2(S)}{S^2} < \infty \tag{2.4}$$

where α is another function which describes a deflection of the σ space V_4 from the pseudo-Euclidean space E_4. The σ space V_4, is a non-Riemannian space that will be referred to as a conformally pseudo-Euclidean σ space. V_4 , will be treated as a model (tubular model) of the real space-time. If $\alpha = 0$, then V_4, coincides with E_4. In this case the timelike tubes degenerate into straight lines, and the tubular model of the space-time is converted into the pseudo-Euclidean space that will be referred to as a linear model. The pseudo-Euclidean space E_4 is a space associated with the σ space V_4. E_4 will not be treated as real space-time. E_4 is an *accessory space* that is used for establishing correspondence between description in V_4, and the conventional description in the pseudo-Euclidean space-time of the special relativity. For instance, if V_4 is a real space-time, then timelike straight lines in E_4 cannot be treated as world lines of real particles. They are only auxiliary constructions that are useful for treating the real world tubes of the tubular model V_4, from the standpoint of the linear one.

Let $\mathbf{P}_0\mathbf{P}_1$, be timelike vector on $\mathbb{M}_4$, and $\mu = \sqrt{2\sigma(P_0, P_1)}$ is its length in V_4, . Then its length μ_{E} in E_4 is

$$\mu_{\mathrm{E}} = \sqrt{2\sigma_{\mathrm{E}}(P_0 P_1)} = g(\mu) = \mu(1 - \alpha(\mu)) \tag{2.5}$$

Using Eqs. (1.6), (1.2), (2.4), (2.1), one can obtain the equation determining the shape of the tube $\mathcal{T}_{P_0P_1}$ in V_4.

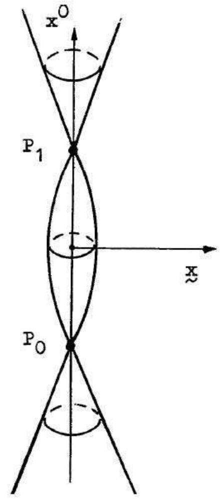

Figure 1: Schematic shape of timelike tube

Let $P_0 = (-\mu_{\rm E}/2, \mathbf{0})$, $P_1 = (\mu_{\rm E}/2, \mathbf{0})$ be coordinates of the points P_0, P_1 in the Galilean coordinate system (2.1)) (2.2). Then the timelike tube $\mathcal{T}_{P_0P_1}$ in V_4, has the shape, presented schematically in Fig. 1. It is a three-dimensional rotation surface that tends asymptotically to the cone surface:

$$\mathbf{x}^2 = \left(x^0\right)^2 \tanh^2\theta, \qquad \tanh\theta = \sqrt{\left(\mu^2 - g^2\left(\mu\right)\right)/g^2\left(\mu\right)} \tag{2.6}$$

Section of the tube by the plane $x^0 = 0$ gives a two-dimensional sphere in E_4 of the radius

$$R_{\rm E} = \sqrt{\frac{1}{4}g^2\left(\mu\right) - g^2\left(\mu/2\right)} \tag{2.7}$$

Thus the shape of the tube $\mathcal{T}_{P_0P_1}$ depends essentially on the length μ of the vector $\mathbf{P}_0\mathbf{P}_1$,. If μ is large enough, and hence $\mu^2 - g^2\left(\mu\right) \ll g^2\left(\mu\right)$, then the angle θ determining aperture of the cone, and the ratio $R_{\rm E}/\mu$ are small, and the tube $\mathcal{T}_{P_0P_1}$distinguishes slightly from the straight $\mathbf{x} = 0$, describing the tube $\mathcal{T}_{P_0P_1}$ in E_4. The less the length μ of the vector $\mathbf{P}_0\mathbf{P}_1$, the more aperture angle of the cone. At $\mu \to 0$ the asymptotic cone tends to the light cone, because according to Eq. (2.4) $\tanh\theta = 1$ at $\mu = 0$.

In classical mechanics, the four-momentum p_i,($p_ip^i = m^2c^2$) given at the time moment t *determines the world line* of the free particle. It is a straight line tangent to four-vector p_i, the world line being independent on the particle mass. Let us treat the timelike vector $\mathbf{P}_0\mathbf{P}_1$, as a momentum vector and its length in the tubular model V_4, as a mass of the particle, the tube $\mathcal{T}_{P_0P_1}$ in V_4, being treated as the world tube of free particle. Such a treatment is possible, because the timelike vector $\mathbf{P}_0\mathbf{P}_1$ determines the world tube in V_4.

Let σ_0 be such a scale that $\alpha\left(\sqrt{2\sigma}\right) \lesssim 1$ at $|\sigma| < \sigma_0$ and $\alpha\left(\sqrt{2\sigma}\right) \ll 1$ at $\sigma \gg \sigma_0$. This supposition agrees with Eq. (2.4). Then at the large mass $\mu \gg \sqrt{2\sigma_0}$

the world tube of the particle distinguishes from a straight line slightly. At $\mu = 0$ the world tube coincides with the light cone that associates usually with world lines of massless photons.

In the linear model the mass μ of a free particle is not a geometric characteristic, because it cannot be determined by the shape of the free particle world line. But in the tubular model the mass μ is a geometric characteristic, because the free particle world tube determines the momentum completely including the mass μ. In this case the mass is measured in units of length, and some universal constant b defined by the relation

$$m = b\mu, \qquad [b] = \mathrm{g/cm} \tag{2.8}$$

is necessary. Here, m is the mass measured in grams, and μ is the mass measured in centimeters. In the linear model any two infinitesimally close points P_0', P_1' ; on $\mathcal{T}_{P_0P_1}$, determine the momentum $m\mathbf{P}_0'\mathbf{P}_1'/\left|\mathbf{P}_0'\mathbf{P}_1'\right|$ (at given mass m). For instance, if $P_0', P_1' \in \mathcal{T}_{P_0P_1}$, then these points determine the world line $\mathcal{T}_{P_0'P_1'} = \mathcal{T}_{P_0P_1}$. The momentum $m\mathbf{P}_0'\mathbf{P}_1'/\left|\mathbf{P}_0'\mathbf{P}_1'\right|$ can be ascribed to the point P_0', because $P_1' \to P_0'$.

In the tubular model two points $P_0', P_1' \in \mathcal{T}_{P_0P_1}$ $(S(P_0', P_1') = S(P_0, P_1) = \mu)$ determine, generally, another world tube $\mathcal{T}_{P_0'P_1'}$ which does not coincide with $\mathcal{T}_{P_0P_1}$. In other words, a measurement of momentum on the tube $\mathcal{T}_{P_0P_1}$ changes the world tube (converts $\mathcal{T}_{P_0P_1}$ into $\mathcal{T}_{P_0'P_1'}$). This fact associates with the measurement in quantum mechanics, where the measurement changes the state of the system. Besides, the momentum $\mathbf{P}_0\mathbf{P}_1$ can be ascribed only to the tube segment $\mathcal{T}_{[P_0P_1]}$, but not to the point P_0. This fact associates with the quantum mechanics, where the particle momentum cannot be measured instantaneously.

Now let us try to use another approach. Let the world tube of a particle of the mass μ be described as a broken tube

$$\mathcal{T}_{\mathrm{br}}(\mu) = \bigcup_i \mathcal{T}_{[P_iP_{i+1}]}, \qquad S(P_i, P_{i+1}) = \mu, \qquad i = 0, \pm 1, \pm 2, ... \tag{2.9}$$

The segments $\mathcal{T}_{[P_iP_{i+1}]}$ are links of the broken tube. In order for the particle to be free, the broken tube $\mathcal{T}_{\mathrm{br}}(\mu)$ should satisfy some conditions. In the case of linear model and infinitesimally short links these conditions reduce to the first Newton law

$$\frac{dp_i}{d\tau} = 0, \qquad i = 0, 1, 2, 3 \tag{2.10}$$

where p_i is the particle momentum in Galilean frame, τ is some parameter along world line.

In the case of finite links Eq. (2.10) can be rewritten in the form

$$\mathbf{P}_i\mathbf{P}_{i-1} \downarrow\uparrow \mathbf{P}_i\mathbf{P}_{i+1}, \qquad i = 0, \pm 1, \pm 2, ... \tag{2.11}$$

Equation (2.11) can be used in the case when the links of the broken world line are not infinetesimally short and satisfy Eq. (2.9). By means of Eqs. (1.3), (1.4), (1.6), (2.9), the relations (2.11) are reduced to the form

$$S(P_{i-1}, P_{i+1}) = 2\mu, \qquad i = 0, \pm 1, \pm 2, ... \tag{2.12}$$

In the linear model Eqs. (2.9)) (2.12) are equivalent to Eq.(2.10) and any link $\mathcal{T}_{[P_iP_{i+1}]}$ determines unambiguously the whole broken tube $\mathcal{T}_{\mathrm{br}}(\mu)$ which is a straight line. Thus Eqs. (2.9), (2.12) can be considered as a wording of the first Newton law which is supposed to be valid for any tubular model of the space-time also.

In the general case the fixed momentum $\mathbf{P}_0\mathbf{P}_1$, and Eqs.(2.9), (2.12) do not determine unambiguously the broken tube $\mathcal{T}_{\mathrm{br}}(\mu)$. It means that the broken tube $\mathcal{T}_{\mathrm{br}}(\mu)$ should be considered as a random tube. Statistical methods should be used for its description.

3 Statistical description

The statistical description we shall use is the conventional description in terms of the statistical ensemble. But it contains some nonconventional details. Statistical descriptions of classical stochastic systems and quanta1 ones distinguish. The first is based on the concept of probability and the second is based on the concept of the probability amplitude. But statistical descriptions of both types of dynamical systems have common features that can be presented without using concepts of probability or probability amplitude.

Such a description is based on the concept of the statistical ensemble as a dynamical system.

Definition 3.1: A dynamical system $\mathcal{S}$ whose state X evolves according to some dynamical equations is a deterministic dynamical system.

Definition 3.2: A dynamical system $\mathcal{S}$ is a nondeterministic dynamical system (or stochastic system), if there exist no dynamical equations that determine its state evolution.

A characteristic property of stochastic system is an irreproducibility of measurements. It means that repeating measurements of the same quantity $\mathcal{R}$ in the same state X of the stochastic system $\mathcal{S}$ give different values R_1, R_2,... . Practically one cannot study stochastic systems without reducing them to deterministic ones. The way of such reduction is determined by the statistical principle.

Statistical principle: A set $\mathcal{S}$ of N ($N\rightarrow\infty$) like independent dynamical systems $\mathcal{S}$ (stochastic or not) is a deterministic dynamical system. This system is called the statistical ensemble. The systems $\mathcal{S}$ are called the ensemble elements.

A result of the measurement of the quantity $\mathcal{R}$ in the ensemble $\mathcal{E}$ is a distribution $f(R)$ which is reproducible at other measurements, even though the measurement of $\mathcal{R}$ in a single ensemble system $\mathcal{S}$ is irreproducible.

All attributes of the dynamical system: energy, momentum, angular momentum, their densities, Lagrangian, and other dynamical quantities can be ascribed to the ensemble $\mathcal{E}$.

The dynamical equations of the ensemble $\mathcal{E}$ are insensitive to the number N of the ensemble elements. Using independence of dynamical equations on N, formally one can set $N=1$ and use the ensemble consisting of one element. Such a procedure will be referred to as an ensemble projection onto one system. Any additive quantity

$\mathcal{R}$ (energy, momentum, etc.) of the ensemble $\mathcal{E}$ "consisting of one element" can be considered as the mean value of this quantity $\mathcal{R}$ for the stochastic system $\mathcal{S}$.

As far as any statistical ensemble $\mathcal{E}'$ is a dynamical system, it can be an element of other ensemble $\mathcal{E}$. Let us consider a Hamiltonian system $\mathcal{S}$ whose state is described by canonical variables $(\mathbf{x}, \mathbf{p})$ and a hierarchy of ensembles that can be constructed on the base of this system $\mathcal{S}$. The state of the general ensemble $\mathcal{E}$ consisting of systems $\mathcal{S}$ is described by a non-negative function $F(\mathbf{x}, \mathbf{p})$. The distribution function F evolves according to the Liouville equation

$$\mathcal{E}: \qquad \frac{\partial F}{\partial t}+\frac{\partial H}{\partial \mathbf{p}}\frac{\partial F}{\partial \mathbf{x}}-\frac{\partial H}{\partial \mathbf{x}}\frac{\partial F}{\partial \mathbf{p}}=0 \tag{3.1}$$

where $H = H(\mathbf{x}, \mathbf{p})$ is the Hamiltonian function of $\mathcal{S}$. The special (pure) statistical ensemble $\mathcal{E}_{\rho,S}$ whose state is described by functions ρ, S given in the coordinate space is an important special case of the statistical ensemble $\mathcal{E}$. In this case

$$F_{\rho,S}\left(\mathbf{x}, \mathbf{p}\right)=\rho\left(\mathbf{x}\right)\delta\left(\mathbf{p}-\boldsymbol{\nabla} S\left(\mathbf{x}\right)\right) \tag{3.2}$$

Substitution of Eq. (3.2) into Eq. (3.1) leads to equations describing the state ρ, S evolution of the pure ensemble $\mathcal{E}_{\rho,S}$

$$\mathcal{E}_{\rho,S}: \left\{ \begin{array}{c} \frac{\partial \rho}{\partial t}+\boldsymbol{\nabla}\left(\rho\frac{\partial H}{\partial \mathbf{p}}\left(\mathbf{x}\boldsymbol{\nabla} S\right)\right)=0 \\ \frac{\partial S}{\partial t}+H\left(\mathbf{x}\boldsymbol{\nabla} S\right)=0 \end{array} \right. \tag{3.3}$$

Equations (3.1) and (3.3) have the system of Hamiltonian equations

$$\mathcal{S}: \qquad \frac{d\mathbf{x}}{dt}=\frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt}=-\frac{\partial H}{\partial \mathbf{x}} \tag{3.4}$$

as characteristics.

The ensemble $\mathcal{E}_{\rho,S}$ described by the system (3.3) will be referred to as a pure ensemble, and the ensemble $\mathcal{E}$ described by Eq. (3.1) will be referred to as mixed one. These terms are taken from the quantum mechanics, where the pure ensemble is described by the wave function ψ. The mixed ensemble is such an ensemble whose elements are pure ensembles. (The mixed ensemble consists of pure ensembles.) In the given case the ensemble $\mathcal{E}_{\rho,S}$ can be described by means of the wave function

$$\psi=\sqrt{\rho}\exp\left(iS/\hbar\right) \tag{3.5}$$

where $\hbar$ is the Planck's constant. The dynamical equation for ψ can be obtained from Eqs. (3.3), (3.5). It is not linear, generally speaking. Here, the Planck's constant is used as a universal constant having dimensionality of action. Its use in the classical statistical ensemble description does not produce any quantum effects. The ensemble $\mathcal{E}$ described by Eq. (3.1) can be treated as a mixed one, because it can be considered as consisting of elements $\mathcal{E}_{\rho,S}$, i.e., of pure ensembles describing by the wave function (3.5). Thus the dynamical systems $\mathcal{S}$, $\mathcal{E}_{\rho,S}$, $\mathcal{E}$ form a hierarchy: $\mathcal{S}$ is an element of the ensemble $\mathcal{E}_{\rho,S}$, $\mathcal{E}_{\rho,S}$ is an element of the ensemble $\mathcal{E}$.

Before considering the statistical description of broken world tubes $\mathcal{T}_{\mathrm{br}}$ let us investigate a pure statistical ensemble $\mathcal{E}_{\rho,S}$ of relativistic classical particles in E_4.

The ensemble $\mathcal{E}_{\rho,S}$ is a Hamiltonian system whose state is described by canonically conjugate variables $\rho(\mathbf{x}), S(\mathbf{x})$ numerated by spatial coordinates $\mathbf{x}$. Its Hamiltonian functional

$$\mathcal{H}(\rho, S) = \int \rho c\sqrt{m^2c^2 + (\mathbf{\nabla} S)^2}d\mathbf{x} \tag{3.6}$$

generates dynamical equations

$$\frac{\partial \rho(\mathbf{x})}{\partial t} = \frac{\delta \mathcal{H}}{\delta S(\mathbf{x})}, \qquad \frac{\partial S(\mathbf{x})}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho(\mathbf{x})}, \tag{3.7}$$

where $\delta/\delta\rho, \delta/\delta S$ are variational derivatives.

Let us note the following mathematical fact. Replacing m^2 by

$$m_q^2 = m^2 + \left(\frac{\hbar}{2c}\mathbf{\nabla}\log\rho\right)^2, \qquad m^2 = \text{const}, \qquad \left(\frac{\hbar}{2c}\mathbf{\nabla}\log\rho\right)^2 \ll m^2 \tag{3.8}$$

in Eq. (3.6)) one obtains dynamical equations that are equivalent in the nonrelativistical approximation to the Schrödinger equation for a free particle

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\mathbf{\nabla}^2\psi \tag{3.9}$$

with the wave function

$$\psi = \sqrt{\rho}\exp\left(\frac{i}{\hbar}\left(mc^2t + S\right)\right) \tag{3.10}$$

Here, the first term mc^2t in exponent compensates the rest mass term of the relativistical variable S.

Finally, let us produce transformation to "geometrical variables"

$$S = bc\tilde{S}, \quad m = b\mu, \quad t = \tau/c, \quad \hbar = bc\sigma_0, \quad \sigma_0 = \text{const} \tag{3.11}$$

where b is defined by Eq. (2.8). Then the Hamiltonian of the quantum particles ensemble takes the form

$$\mathcal{H}\left(\rho, \tilde{S}\right) = \int \sqrt{\mu_q^2 + \left(\mathbf{\nabla}\tilde{S}\right)^2}\rho d\mathbf{x} \tag{3.12}$$

$$\mu_q^2 = \mu^2 + \left(\frac{\sigma_0}{2}\mathbf{\nabla}\log\rho\right)^2 \tag{3.13}$$

One can see from Eq. (3.12) that quantum effects arise on account of dependence of the mass μ on the ensemble state (ρ, S).

Unlike the Hamiltonian (3.6) the Hamiltonian (3.12) cannot be considered as a sum of Hamiltonians of independent systems, because the mass μ_q depends on the ensemble state ρ, S. It means that in the hierarchy $\mathcal{S}$, $\mathcal{E}_{\rho,S}$, $\mathcal{E}$ the state of a single system $\mathcal{S}$ does not satisfy any dynamical equation, i.e., $\mathcal{S}$ is a stochastic system. The dynamical system $\mathcal{E}_{\rho,S}$ is the first deterministic dynamical system in the hierarchy $\mathcal{S}$, $\mathcal{E}_{\rho,S}$, $\mathcal{E}$. Investigation of the stochastic system $\mathcal{S}$ is meaningless. The first dynamical system on the hierarchy that has to be investigated is the pure ensemble $\mathcal{E}_{\rho,S}$.

4 Oriented mass

Definition 4.2: The oriented mass $\mu_o(P)$ of the tube $\mathcal{T}_{P_0P_1}$ in V_4 is defined by the equation

$$\mu_o = \frac{(\mathbf{P}_0\mathbf{P}_1 \cdot \mathbf{P}_0\mathbf{P})}{|\mathbf{P}_0\mathbf{P}_1|} = \begin{cases} \mu & P \in \mathcal{T}_{[P_0P_1} \\ -\mu & P \in \mathcal{T}_{P_0]P_1} \end{cases}, \quad \mu = S(P_0, P_1) \tag{4.1}$$

In the case of the linear model $\mu_o(P)$ degenerates into

$$\mu_o = p_i u^i = \pm\mu \tag{4.2}$$

where $p_i = \mathbf{P}_0\mathbf{P}$ and $u^i = \mathbf{P}_0\mathbf{P}_1 / |\mathbf{P}_0\mathbf{P}_1|$ are correspondently four-momentum and four-velocity of the particle. The sign of the mass describes the mutual orientation of four-momentum and four-velocity. It is different for particle and antiparticle.

Definition 4.2: The associated oriented mass μ_{oE} of $\mathcal{T}_{P_0P_1}$ in V_4 is the oriented mass of $\mathcal{T}_{P_0P_1}$ considered from the standpoint of the associated σ space E_4 :

$$\mu_{oE} = \frac{(\mathbf{P}_0\mathbf{P}_1 \cdot \mathbf{P}_0\mathbf{P})_\mathrm{E}}{|\mathbf{P}_0\mathbf{P}_1|_\mathrm{E}}, \qquad P \in \mathcal{T}_{[P_0P_1}, \tag{4.3}$$

where index "E" means that the scalar product is calculated in the associated space E_4. Vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}$, $P \in \mathcal{T}_{[P_0P_1}$ are not collinear in E_4. Vector of four-momentum and that of four-velocity are not collinear in E_4.

In the coordinate system, where spatial components of the four-momentum vanish,

$$p_i = (\mu_\mathrm{E}, \mathbf{0}). \qquad u^i = \left(1/\sqrt{1-\boldsymbol{\beta}^2}, \boldsymbol{\beta}/\sqrt{1-\boldsymbol{\beta}^2}\right) \tag{4.4}$$

one has

$$\mu_{o\mathrm{E}} = p_i u^i = \mu_\mathrm{E}/\sqrt{1-\boldsymbol{\beta}^2} \simeq \mu_\mathrm{E}\left(1 - \boldsymbol{\beta}^2/2\right), \qquad \boldsymbol{\beta}^2 \ll 1 \tag{4.5}$$

Here $\boldsymbol{\beta}$ is a velocity in the coordinate system, where $\mathbf{p} = 0$, and $\theta = \mathrm{arctanh}\beta$ is the angle between vectors p_i and u^i

For the broken world tube (2.9) of the mass μ the oriented associated mass is defined as follows

$$\mu_{o\mathrm{E}}(P) = \frac{(\mathbf{P}_i\mathbf{P}_{i+1} \cdot \mathbf{P}_i\mathbf{P})_\mathrm{E}}{|\mathbf{P}_i\mathbf{P}|_\mathrm{E}}, \qquad P \in \mathcal{T}_{(P_iP_{i+1}]} \subset \mathcal{T}_\mathrm{br}(\mu) \tag{4.6}$$

5 Determination of mapping function f

Let us consider a nonrelativistic ensemble $\mathcal{E}$ of broken tubes $\mathcal{T}_\mathrm{br}(\mu)$ in the tubular model V_4, . Let this ensemble be described in the associated space E_4. It is supposed that among the ensembles $\mathcal{E}$ there are pure nonrelativistic ensembles $\mathcal{E}_{\rho,S}$ having the following properties.

(1) The state of the pure ensemble $\mathcal{E}_{\rho,S}$ is described in E_4 by the particle concentration ρ and momentum $p_i = \left\{ \sqrt{\mu_{\mathrm{E}}^2 + (\boldsymbol{\nabla} S)^2}, \boldsymbol{\nabla} S \right\}$, $(p_i p^i = \mu_{\mathrm{E}}^2)$.

(2) The ensemble $\mathcal{E}_{\rho,S}$ state evolution is described by the Hamiltonian functional (3.12)) where μ_q coincides with the oriented associated mass $\mu_{o\mathrm{E}}$ which is some function of the ensemble state (ρ, S). $\mu_{o\mathrm{E}}$ is defined by Eqs. (4.4)) (4.5). $\boldsymbol{\beta}$ is the mean velocity at the point P in the coordinate system, where the mean momentum $\mathbf{p}$ vanishes.

The $\boldsymbol{\beta}$ is some function of the ensembIe state (ρ, S). It must be calculated basing on the tubular space-time model V_4. That mode1 V_4, where

$$\boldsymbol{\beta} = -\frac{\sigma_0}{2\mu_{\mathrm{E}}} \frac{\boldsymbol{\nabla}\rho}{\rho} \tag{5.1}$$

and, hence,μ_{oE}^2 coinciding with μ_q^2 in Eq. (3.13) is an optima1 one, because the ensemble $\mathcal{E}_{\rho,S}$ of broken tubes coincides with quanta1 ensemble (3.12), and the tubes $\mathcal{T}_{\mathrm{br}}(\mu)$ of this V_4, can be treated as world tubes of real particles. Let us note that Eq. (5.1) coincides with the expression for the mean velocity of the ensemble of Brownian particles, where particle concentration is ρ and diffusion coefficient is $D = 0.5\sigma_0/\mu_{\mathrm{E}}$.

Let us calculate the mean velocity $\boldsymbol{\beta}$ in the ensemble $\mathcal{E}_{\rho,S}$. Let us attribute the momentum $\mathbf{p}$, defined by the vector $\mathbf{P}_0\mathbf{P}_1$, to the point P which is placed in the middle of the segment $[P_0, P_1]$ in E_4 (see Fig. 2):

$$S_{\mathrm{E}}(P_0, P) = S_{\mathrm{E}}(P, P_1) = \frac{1}{2} S_{\mathrm{E}}(P_0, P_1) = \frac{1}{2} g(\mu) = \frac{1}{2}\mu_{\mathrm{E}} \tag{5.2}$$

It is supposed that the tube segment $\mathcal{T}_{[P_0' P_1']}$ of any broken tube which contains the point P, $P \in \mathcal{T}_{[P_0' P_1']}$ contributes into the value $u^i(P)$ of the four-velocity at P. This contribution is proportional to $x^i(P) - x^i(P')$, where $x^i(Q)$ are Galilean coordinates of the point Q in E_4. This coordinate system is supposed to be chosen in such a way, that

$$x^i(P) = 0, \qquad p^i = x^i(P_1) - x^i(P_0) = (\mu_{\mathrm{E}}, \mathbf{0}) \tag{5.3}$$

For simplicity, contributions of only those segments $\mathcal{T}_{[P_0' P_1']}$ are taken into account for which

$$S(P, P_0') = S(P, P_1'), \qquad P \in \mathcal{T}_{[P_0' P_1']} \tag{5.4}$$

It means that in E_4 the point $P \in \mathcal{T}_{[P_0' P_1']}$ is maximally remoted from the segment $[P_0' P_1']$. Let P' be the middle of the segment $[P_0' P_1']$ in E_4, i.e.,

$$S_{\mathrm{E}}(P_1', P') = S_{\mathrm{E}}(P', P_0') = \frac{1}{2} S_{\mathrm{E}}(P_0', P_1') = \frac{1}{2} g(\mu) = \frac{1}{2}\mu_{\mathrm{E}} \tag{5.5}$$

then

$$S_{\mathrm{E}}(P, P') = i R_{\mathrm{E}} \sqrt{\frac{1}{4} g^2(\mu) - g^2(\mu/2)} \tag{5.6}$$

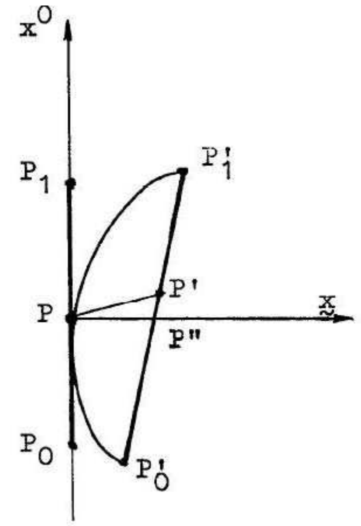

Figure 2: Momentum p_i, defined by the vector $\mathbf{P}_0\mathbf{P}_1$ to the point P which is placed in the middle of the segment $[P_0P_1]$ in E_4

The flux density $j^i(P'')$ is defined by the relation

$$j_k(P'') = \left(\rho\left(P''\right), \frac{\rho\left(P''\right)\boldsymbol{\nabla} S\left(P''\right)}{\sqrt{\mu_{\mathrm{E}}^2 + \left(\boldsymbol{\nabla} S\left(P''\right)\right)^2}}\right) \tag{5.7}$$

and vector $j^i(P')$ is parallel to $p^i(P')$ in E_4.

Let

$$y^i = x^i\left(P''\right), \qquad i = 0,1,2,3; \qquad y^0 = 0 \tag{5.8}$$

The flux density $j^i(y)$ is supposed to change on the characteristic length $L = \mu/\varepsilon$, $\varepsilon \ll 1$. Then

$$j^i\left(P''\right) = j^i\left(P\right) + j^i_{,k}\left(P\right)y^k + O\left(\varepsilon^2\right), \quad j^i_{,k} \equiv \partial_k j^i, \quad \mu\partial_k j^i/j^0 = O\left(\varepsilon\right) \tag{5.9}$$

One can calculate that

$$x^i\left(P'\right) = y^i + q^i\left(P'\right)l_{\mathrm{E}}, \quad l_{\mathrm{E}} = S_{\mathrm{E}}\left(P', P''\right) = \sqrt{R_{\mathrm{E}}^2 - \mathbf{y}^2}$$

$$q^i\left(P'\right) \equiv \frac{j^i\left(P'\right)}{\left|j\left(P'\right)\right|} = \frac{j^i\left(P\right) + j^i_{,k}\left(P\right)\left(y^k + \delta_0^k l_{\mathrm{E}}\right)}{j^0\left(P\right)} + O\left(\varepsilon^2\right) \tag{5.10}$$

Calculation shows that Eqs. (5.4), (5.5) are equivalent to

$$R_{\mathrm{E}}^2 - \mathbf{y}^2 + O\left(\varepsilon^2\right) = 0 \tag{5.11}$$

Coordinates $x^i(P_0')$ are determined by the relation

$$\begin{aligned} x^i(P_0') &= y^i - q^i\left(P'\right)\left(S_{\mathrm{E}}\left(P_0', P'\right) - l_{\mathrm{E}}\right) \\ &= y^i - \frac{j^i + j^i_{,\alpha}y^\alpha + j^i_{,0}l_{\mathrm{E}}}{j^0}\left(\frac{g\left(\mu\right)}{2} - l_{\mathrm{E}}\right) \end{aligned} \tag{5.12}$$

Let us define the mean four-velocity by means of the expression

$$u^i(P) = A\int \left(x^i(P) - x^i(P_0')\right) j^i(P'')\,\delta\left(R_{\mathrm{E}}^2 - \mathbf{y}^2\right) d\mathbf{y} + O\left(\varepsilon^e\right) \tag{5.13}$$

where the normalization factor A is defined by the relation

$$u^i(P)\,u_i(P) = 1 \tag{5.14}$$

One obtains

$$\begin{aligned} u^0 &= A\int \frac{1}{2} g(\mu)\left(j^0 + j^0_{,\alpha} y^\alpha\right)\delta\left(R_{\mathrm{E}}^2 - \mathbf{y}^2\right) d\mathbf{y} \\ &= \pi R_{\mathrm{E}}^2 A g(\mu)\, j^0 + O\left(\varepsilon^2\right) \end{aligned} \tag{5.15}$$

$$\begin{aligned} u^\alpha &= A\int \left(-y^\alpha\right)\left(j^0 + j^0_{,\gamma} y^\gamma\right)\delta\left(R_{\mathrm{E}}^2 - \mathbf{y}^2\right) d\mathbf{y} \\ &= -\frac{2\pi}{3} A R_{\mathrm{E}}^3 j^0_{,\alpha} + O\left(\varepsilon^2\right) \end{aligned} \tag{5.16}$$

It follows from Eqs. (5.15), (5.16)

$$\boldsymbol{\beta} = \frac{\mathbf{u}}{u^0} = -\frac{2}{3}\frac{R_{\mathrm{E}}^2}{\mu_{\mathrm{E}}}\frac{\boldsymbol{\nabla}\rho}{\rho} \tag{5.17}$$

Comparing Eqs. (5.1) and (5.17), one obtains by means of Eq. (2.7)

$$R_{\mathrm{E}}^2 = \frac{1}{2} g^2(\mu) - g^2\left(\frac{\mu}{2}\right) = \frac{3}{4}\sigma_0 \tag{5.18}$$

Let Eq. (5.18) take place for any mass $\mu > \mu_e$, where μ_e is the electron mass, i.e., the mass of the lightest massive particles. If the particle mass $\mu = 0$, then the nonrelativistic approximation which was used for obtaining Eq. (5.18) does not exist. One has a solution of the functional equation (5.18):

$$\mu_{\mathrm{E}}^2 = g^2(\mu) = \mu^2 - \sigma_0, \qquad \mu \geq \mu_e \tag{5.19}$$

It follows from Eqs. (2.3), (5.19)

$$\sigma = \sigma_{\mathrm{E}} + \frac{\sigma_0}{2}, \qquad \sigma_0 = \frac{\hbar}{bc}, \qquad \sigma \geq \frac{\mu_e^2}{2} \tag{5.20}$$

On the other hand, according to Eq. (1.1), σ and σ_{E} vanish simultaneously. Substituting $\mu = \mu_e$ into Eq. (5.19) and using the relation $\mu_{\mathrm{E}}^2 = g^2(\mu_e) > 0$, one obtains

$$\mu_e > \sigma_0 \tag{5.21}$$

It follows from Eqs. (5.21), (3.11)

$$b < 10^{-16.5}\mathrm{g/cm}, \qquad \sqrt{\sigma_0} < 10^{-10.5}\mathrm{cm} \tag{5.22}$$

The maximal section radius of the broken tube

$$R_{\mathrm{E}} - \frac{\sqrt{3\sigma_0}}{2} < 10^{-10.4}\mathrm{cm} \tag{5.23}$$

is the same for all massive particles. The quantity $\sqrt{\sigma_0}$ can be treated as an elementary length. The obtained relation (5.20) determines the optimal tubular model of the space-time. This space-time model is responsible for quantum effects.

Let us note the relation (5.20) determines the mapping function f only for the values of argument $\sigma_{\mathrm{E}} \geq (\mu_e^2 - \sigma_0)/2 > 0$.For $\sigma_{\mathrm{E}} < (\mu_e^2 - \sigma_0)/2$ the mapping function f remains indeterminate.

6 Discussion

The tubular model (5.20) is a more attractive model of the space-time than the linear one, because it explains some more quantum properties. In particular, it permits random world tubes to exist, the pure ensemble of these tubes being described by the Schrödinger equation.

Of course, the quantum mechanics principIes are not exhausted by the Schrödinger equation. Besides the linearity principle they include more the rule of computation of the physical quantities average values. Using the statistical principle [7, 8] this rule can be obtained. According to the statistical principle the statistical ensemble is a dynamical system consisting of many dynamical systems or stochastical ones. At a proper normalization all additive physical quantities (energy, momentum, angular momentum) of the ensemble can be considered as the mean values of the corresponding quantities for the ensemble elements. Besides, for the nonrelativistical ensemble the ρ can be treated as a probability density of the particle position. It allows us to calculate the mean values of the type $F(\mathbf{x})$. As regards to mean values of type $F(\mathbf{p})$, $F(\mathbf{x}, \mathbf{p})$ one cannot calculate them for the pure ensemble (3.6)-(3.8). In the conventional quantum mechanics the value $F(\mathbf{p})$ can be calculated, but the truth of this calculated value $F(\mathbf{p})$ cannot be tested experimentally [9], because the momentum cannot be measured instantly. The system state ψ changes during the measuring time, and the measured value cannot be ascribed to any definite state ψ. But the quantum mechanics always ascribes any measured value to some definite state ψ.

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