Inadequacy of the linear vector space formalism at metric approach to geometry

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Abstract

It is shown that formalism of linear vector space is inadequate at the metric approach to geometry, when geometry is described completely in terms of the distance function d, or in terms of the world function $\sigma = d^2/2$. Operations of the linear vector space appear to be ambiguous, if they are introduced at the metric approach to geometry.

Key words: metric approach; deformation principle; discrete geometry; world function; geometry with indefinite dimension;

There are two approaches to geometry: (1) physical approach and (2) mathematical approach. At the physical approach the geometry is called the physical geometry. It is considered as a science on properties of space or on properties of space-time. The physical geometry can be formulated in a coordinateless form. The main objects of the physical geometry are points (events) of the space-time, space-time distances between the points and geometrical objects constructed of the space-time points. In particular, in the physical geometry the vector $\mathbf{PQ}$ is the ordered set $\{P, Q\}$ of two points $P, Q \in \Omega$, where Ω is the point set, where the geometry is given. The vector $\mathbf{PQ}$ will be referred to as the geometrical vector (g-vector).

At the mathematical approach the geometry is considered as an abstract logical construction where the vector $u \in \mathcal{L}_n$ is an element of linear vector space $\mathcal{L}_n$. We shall refer to the mathematical approach to geometry as the mathematical geometry. There are operations of the linear vector space (summation of vectors and multiplication of a vector by a real number). These operations obey the linear vector space axioms. Generally speaking, vectors $u \in \mathcal{L}_n$ do not coincide with g-vectors of space-time, and we shall refer to vectors $u \in \mathcal{L}_n$ as linear vectors (linvectors). In the proper Euclidean geometry and in some other cases the linvectors can be identified with g-vectors. Then operations of the linear vector field $\mathcal{L}_n$ can be applied to g-vectors of Euclidean geometry and of the Minkowski geometry. In this case the mathematical geometry coincide with the physical geometry.

In the linear vector space (i.e. in the mathematical geometry) one can always introduce a coordinate system and decompose any vector along basic vectors of the coordinate system. In the physical geometry such a decomposition is not always possible.

Primordial meaning of geometry was physical. Euclid presented his geometry as a logical construction. He applied geometry for description of the real space properties, but he did not use a coordinate system, and he did not use linear vector space. Introduction of the coordinate system to the Euclidean geometry was conditioned by needs of classical mechanics. This circumstance generated application of the linear vector space in the physical geometry, i.e. in the science describing properties of the real space.

Thus, the mathematical geometry (i.e. the logical construction with its linear vector space) has been used for description of the real space. Thereafter one considered that the linear vector space is an attribute of the space-time geometry. Now the accepted viewpoint contains the statement, that the linear vector field can be introduced (maybe, locally) in any space-time geometry, and the space-time geometry is a mathematical geometry. In particular, the symplectic geometry, which has nothing to do with the space-time geometry, is considered nevertheless as a geometry (mathematical), because its formalism coincides with the formalism of the Euclidean geometry in the coordinate presentation (one can introduce linear vector space in the symplectic geometry).

In this paper we investigate at which conditions one can introduce linear vector space in the space-time geometry. Or at which conditions can one use the mathematical geometry in description of the space-time?Any generalized geometry $\mathcal{G}$ is some generalization of the proper Euclidean geometry $\mathcal{G}_{\rm E}$. We are interested only in physical geometries, which enable to describe the space-time geometry. The physical geometry is described in the framework of the metric approach to geometry, when the point set Ω of space-time points (events) is described completely by the distance function $d(P,Q)$, $P,Q \in \Omega$, or by the world function $\sigma = \frac{1}{2}d^2$. The distance d can be imaginary for a spacelike distance between the points $P,Q \in \Omega$, whereas the world function σ is always real.

The Euclidean geometry is a degenerate geometry in the sense that a generalized geometry $\mathcal{G}$ may have such properties, which are absent in the proper Euclidean geometry $\mathcal{G}_{\rm E}$. For instance, the straight line segment $\mathcal{T}_{[PQ]}$ between the points P,Q may be not one-dimensional in $\mathcal{G}$, although it is always one-dimensional in $\mathcal{G}_{\rm E}$. Besides, equality of two vectors $\mathbf{PQ}$ and $\mathbf{RS}$ may be multivariant. It means that at the point P there are many vectors $\mathbf{PQ}$, $\mathbf{PQ}'$, $\mathbf{PQ}''$,... which are equivalent to the vector $\mathbf{RS}$ at the point R, but vectors $\mathbf{PQ}$, $\mathbf{PQ}'$, $\mathbf{PQ}''$,...are not equivalent between themselves.

The way of generalization of $\mathcal{G}_{\rm E}$ depends on the way of representation of $\mathcal{G}_{\rm E}$ [1].

As far as the generalized geometry may be discrete, one should use for generalization only σ-representation of $\mathcal{G}_{\mathrm{E}}$, when the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is considered as a physical geometry, i.e. it is described in terms and only in terms of the Euclidean world function σ_{E}. It is necessary because $\mathcal{G}_{\mathrm{E}}$ and a discrete geometry $\mathcal{G}_{\mathrm{d}}$ have a unique common concept: the distance.

The physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ is defined on arbitrary point set Ω, where σ is a single-valued function, defined as

$$\sigma: \quad \Omega \times \Omega \rightarrow \mathbb{R}, \quad \sigma(P,Q) = \sigma(Q,P), \quad \sigma(P,P) = 0, \quad \forall P,Q \in \Omega \tag{1}$$

The discrete geometry $\mathcal{G}_{\mathrm{d}}$ is restricted by the relation

$$|d(P,Q)| \equiv \left|\sqrt{2\sigma_{\mathrm{d}}(P,Q)}\right| \notin (0, \lambda_0), \quad \forall P,Q \in \Omega \tag{2}$$

where $\lambda_0 > 0$ is the elementary length, which is a parameter of $\mathcal{G}_{\mathrm{d}}$. The restriction (2) is condition on world function σ_{d} (not on the set Ω, which is an arbitrary set of points). In particular, the discrete geometry $\mathcal{G}_{\mathrm{d}} = \{\sigma_{\mathrm{d}}, \Omega_{\mathrm{M}}\}$ can be given on the point set Ω_{M}, where the geometry of Minkowski $\mathcal{G}_{\mathrm{M}} = \{\sigma_{\mathrm{M}}, \Omega_{\mathrm{M}}\}$ is given. The geometry $\mathcal{G}_{\mathrm{d}}$ is obtained from $\mathcal{G}_{\mathrm{E}}$ by a replacement of the Euclidean world function σ_{E} by the world function σ_{d} in all definitions of the geometry $\mathcal{G}_{\mathrm{E}}$ written in terms of the σ_{E}.

The condition (2) is satisfied, if σ_{d} is taken, for instance, in the form

$$\sigma_{\mathrm{d}} = \sigma_{\mathrm{M}} + \frac{\lambda_0^2}{2}\operatorname{sgn}(\sigma_{\mathrm{M}}) \tag{3}$$

where σ_{M} is the world function of the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$ and

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 1 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 1 \end{cases} \tag{4}$$

The point set Ω_{d} is the same as in $\mathcal{G}_{\mathrm{M}}$.

In $\mathcal{G}_{\mathrm{E}}$ one uses the formalism of the linear vector space $\mathcal{L}_n$. A vector $u \in \mathcal{L}_n$ is some abstract quantity, which is defined by its properties. In $\mathcal{L}_n$ operations of summation of vectors and multiplication of a vector by a real number are defined

$$(u+v) \in \mathcal{L}_n, \quad \text{if} \;\; u \in \mathcal{L}_n, v \in \mathcal{L}_n$$

$$au \in \mathcal{L}_n, \quad \text{if} \;\; u \in \mathcal{L}_n, \quad a \in \mathbb{R}$$

In $\mathcal{L}_n$ any vector exists in one copy. There is no equivalent vectors in $\mathcal{L}_n$. In the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}} = \{\sigma_{\mathrm{E}}, \Omega_{\mathrm{E}}\}$ there are many equivalent vectors, because in $\mathcal{G}_{\mathrm{E}}$ vector $\mathbf{PQ} = \{P, Q\}$ is the ordered set of two points P and Q. As far as vectors in $\mathcal{L}_n$ and in $\mathcal{G}_{\mathrm{E}}$ have different properties, we shall use different names for them. Vector $u \in \mathcal{L}_n$ will be referred to as linear vector (linvector), and vector $\mathbf{PQ} \in \mathbf{\Omega} \times \mathbf{\Omega}$ will be referred to as a geometrical vector (g-vector).

Two g-vectors $\mathbf{PQ}$ and $\mathbf{RS}$ are equivalent ($\mathbf{PQ}\text{eqv}\mathbf{RS}$), if their lengths are equal and they are in parallel ($\mathbf{PQ} \upuparrows \mathbf{RS}$)

$$(\mathbf{PQ} \upuparrows \mathbf{RS}): \quad (\mathbf{PQ}.\mathbf{RS}) = |\mathbf{PQ}| \cdot |\mathbf{RS}| \tag{5}$$

Here $(\mathbf{PQ}.\mathbf{RS})$ is the scalar product of two g-vectors $\mathbf{PQ}$ and $\mathbf{RS}$, defined in terms of the world function in the form

$$(\mathbf{PQ}.\mathbf{RS}) = \sigma(P,S) + \sigma(Q,R) - \sigma(P,R) - \sigma(Q,S) \tag{6}$$

The length $|\mathbf{PQ}|$ of the vector $\mathbf{PQ}$ is defined by the relation

$$|\mathbf{PQ}| = \sqrt{2\sigma(P,Q)} \tag{7}$$

Thus, the g-vectors $\mathbf{PQ}$ and $\mathbf{RS}$ are equivalent, if

$$(\mathbf{PQ}\text{eqv}\mathbf{RS}): \quad (\mathbf{PQ}.\mathbf{RS}) = |\mathbf{PQ}| \cdot |\mathbf{RS}| \wedge |\mathbf{PQ}| = |\mathbf{RS}| \tag{8}$$

Let $\Omega_{\mathbf{AB}} \subset \Omega \times \Omega$ be the set of all g-vectors $\mathbf{CD} \in \Omega_{\mathbf{AB}} \subset \Omega \times \Omega$, which are equivalent (equal) to g-vector $\mathbf{AB}$. In $\mathcal{G}_{\mathrm{E}}$ all g-vectors belonging to $\Omega_{\mathbf{AB}}$ are equivalent between themselves, and the set $\Omega_{\mathbf{AB}}$ is an equivalence class $[\mathbf{AB}]$ of the g-vector $\mathbf{AB}$. Linvectors of $\mathcal{L}_n$ can be mapped onto equivalence classes of g-vectors of $\Omega \times \Omega$. Operations of $\mathcal{L}_n$ can be used for construction of the geometrical formalism in $\mathcal{G}_{\mathrm{E}}$. The equivalence relation is transitive in $\mathcal{G}_{\mathrm{E}}$, and it is a reason, why the set $\Omega_{\mathbf{AB}}$ of g-vectors, which are eqivalent to g-vector $\mathbf{AB}$, forms the equivalence class $[\mathbf{AB}]$.

In the discrete geometry (3) the equivalence relation is intransitive, and the set $\Omega_{\mathbf{AB}}$ contains g-vectors, which are not equivalent between themselves. In this case the equivalence of two vectors is multivariant, and the set $\Omega_{\mathbf{AB}}$ of g-vectors, which are equivalent to g-vector $\mathbf{AB}$, does not form the equivalence class $[\mathbf{AB}]$.

Formally one can define the operation of summation of g-vectors in $\mathcal{G}_{\mathrm{d}}$, but it will be ambiguous. Indeed, the sum $\mathbf{AC}$ of two g-vectors $\mathbf{AB}$ and $\mathbf{BC}$, when the end of one g-vector is the origin of other one, is defined as follows

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC} \tag{9}$$

The sum $\mathbf{AD}_1$ of two arbitrary g-vectors $\mathbf{AB}$ and $\mathbf{CD}$ at the point A is defined as follows

$$\mathbf{AB} + \mathbf{CD} = \mathbf{AB} + \mathbf{BD}_1 = \mathbf{AD}_1, \quad (\mathbf{CD}\text{eqv}\mathbf{BD}_1) \tag{10}$$

The g-vector $\mathbf{AD}_1$ is defined by relation (10) ambiguously, because the g-vector $\mathbf{BD}_1$ is determined ambiguously by the equivalence relation $(\mathbf{CD}\text{eqv}\mathbf{BD}_1)$.

The g-vector $\mathbf{AC} = a\mathbf{AB}$, which is a result of multiplication of g-vector $\mathbf{AB}$ by a real number a is defined by the relations

$$a\mathbf{AB} = \mathbf{AC}, \quad |\mathbf{AC}| = a|\mathbf{AB}|, \quad (\mathbf{AB}.\mathbf{AC}) = a|\mathbf{AB}|^2 \tag{11}$$

Result of multiplication is ambiguous, because, generally speaking, the system of two last equations (11) has no unique solution in $\mathcal{G}_{\mathrm{d}}$. In the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ operations (10) and (11) are defined uniquely.

Thus, formalism of the linear vector space $\mathcal{L}_n$ is inadequate in the discrete geometry $\mathcal{G}_{\mathrm{d}}$. This formalism is inadequate in any physical geometry, where the equivalence relation of two g-vectors is intransitive. For instance, in the Riemannian geometry presented in the form of a physical geometry the equivalence of two g-vectors $\mathbf{AB}$ and $\mathbf{CD}$ is multivariant, generally speaking. But the equivalence relation is single-variant, if the origin points A and C coincide ($A = C$). This property is well known as an absence of fern-parallelism in the Riemannian geometry. Usually one tries to suppress the multivariance by additional restriction, generated by the parallel transport. But multivariance of the equivalence relation is a natural property of the Riemannian geometry, and it is hardly reasonable to suppress this multivariance artificially.

Now about terminology. Some mathematicians states that the equivalence relation is transitive by definition, and one may not use the term "equivalence relation" for the relation defined by formulas (8). They say: "One should use another term, the term equivalence is busy." Let we follow this advice and use the term "intr-equivalence" for the relation between g-vectors, defined by (8). Then in $\mathcal{G}_{\mathrm{E}}$, which is a special case of physical geometry, the intr-equivalence turns to transitive equivalence. It means that the transitive equivalence is a special case of intr-equivalence. As far as "intr-equivalence" appears to be a more general concept, than transitive equivalence, one should replace the terms. One should use the shorter term "equivalence" in general case, when the equivalence relation is intransitive, generally speaking, and the longer term "transitive equivalence" should be used in that special case, when the equivalence is transitive. The demand, that the equivalence relation is transitive by definition is conditioned by the fact, that mathematicians dealed before only with the transitive equivalence relation. They believed that equivalence cannot be intransitive.

However, the definition of the g-vector equivalence in the form of (8) is valid in any physical geometry and, in particular, in the discrete space-time geometry $\mathcal{G}_{\mathrm{d}} = \{\sigma_{\mathrm{d}}, \Omega_{\mathrm{M}}\}$, where the world function (3) is given on the manifold of Minkowski Ω_{M}. The discrete space-time geometry $\mathcal{G}_{\mathrm{d}}$ describes the real space-time geometry in microcosm better, than the Riemannian geometry does. But the discrete geometry cannot be constructed on the basis of the linear vector space formalism. The discrete geometry $\mathcal{G}_{\mathrm{d}}$ (and other physical geometries) is constructed by means of the deformation principle [3]. The discrete geometry is obtained as a deformation of the Euclidean geometry $\mathcal{G}_{\mathrm{E}} = \{\sigma_{\mathrm{E}}, \Omega\}$. It means that the world function σ_{E} is replaced in all definitions of $\mathcal{G}_{\mathrm{E}}$ written in terms of Euclidean world function σ_{E} by the world function of $\mathcal{G}_{\mathrm{d}}$. These definitions are definitions of geometrical objects and general geometric concepts.

For instance, in $\mathcal{G}_{\mathrm{E}}$ the definition of the straight segment $\mathcal{T}_{[P_0P_1]}$ between the points P_0 and P_1 has the form

$$\mathcal{T}_{[P_0P_1]} = \left\{ R | \sqrt{2\sigma (P_0, R)} + \sqrt{2\sigma (R, P_1)} - \sqrt{2\sigma (P_0, P_1)} = 0 \right\} \tag{12}$$

where $\sigma = \sigma_{\mathrm{E}}$. In $\mathcal{G}_{\mathrm{d}}$ the straight segment $\mathcal{T}_{[P_0P_1]}$ is described by the same relation (12), but now $\sigma = \sigma_{\mathrm{d}}$. The deformation principle admits one to recognize the same geometrical object in different geometries (at different world functions). It is very important in the space-time geometry, where different regions have different geometries which are described by different world functions. Without the deformation principle such a recognition is impossible. For instance, using the metrical approach and constructing the distance geometry, Blumenthal [4] had not the deformation principle. He can construct a curve only as a continuous mapping $[0, 1] \rightarrow \Omega$. Such a construction of a curve cannot be used in a discrete geometry. Besides, he used the concept of mapping which is not defined at the consequent metric approach, when the geometry is described in terms of distance and only in terms of distance.

In the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ there are general geometrical concepts, which can be formulated in terms of σ_{E}. Besides, there are relations describing special Euclidean properties of the Euclidean world function σ_{E}. Such concepts as equivalence of vectors, scalar product of vectors, linear dependence of vectors are general geometric concepts, which can be formulated in terms of σ_{E}. These concepts can formulated in the same form in the discrete geometry after replacement of σ_{E} by σ_{d}. However, such concepts as a dimension of the geometry and coordinate system contains special properties of the Euclidean world function σ_{E}. The cannot be used in the discrete geometry and other physical geometry. For instance, in the discrete geometry (3) the dimension as a maximal number of linear independent vectors cannot be introduced.

Geometry without of a definite dimension looks rather unexpected, because dimension and a coordinate system are considered as axiomatic quantities. Construction of the Riemannian geometry begins with introduction of a manifold and a coordinate system of definite dimension on it. One does not discuss the question, whether or not an introduction of a coordinate system of definite dimension is possible. The metric approach and the deformation principle admit an introduction of coordinateless formulation of the geometry. It is a worth of the metric approach, which is not used usually in investigation of the space-time geometry. As a result the investigation of the discrete space-time geometry was not developed. The case, when condition of discreteness (2) is considered as a restriction on the properties of the point set Ω leads to a geometry on a lattice, which can hardly be considered as a valuable discrete geometry, especially in its application to the space-time geometry.

Let us formulate general geometric properties of the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. The scalar product of two g-vectors is defined by (6). Equivalence of two g-vectors is defined by (8).

n g-vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$ are linear dependent, if and only if the Gram determinant

$$F_n (\mathcal{P}_n) = \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)|| , \quad i, k = 1, 2, ...n, \quad \mathcal{P}_n \equiv \{P_0, P_2, ...P_n\} \tag{13}$$

vanishes

$$F_n(\mathcal{P}_n) = 0 \tag{14}$$

The special relations of the n-dimensional proper Euclidean geometry have the form [2]:

I. Definition of the metric dimension:

$$\exists \mathcal{P}_n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \qquad F_n(\mathcal{P}_n) \neq 0, \qquad F_k\left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{15}$$

where $F_n(\mathcal{P}_n)$ is the n-th order Gram's determinant (13). g-vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, ...n$ are basic g-vectors of the rectilinear coordinate system K_n with the origin at the point P_0. The covariant coordinates of the point P in the coordinate system K_n are defined by the relation

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i = 1, 2, ...n \tag{16}$$

The metric tensors $g_{ik}(\mathcal{P}_n)$ and $g^{ik}(\mathcal{P}_n)$, $i, k = 1, 2, ...n$ in K_n are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}(\mathcal{P}_n)\, g_{lk}(\mathcal{P}_n) = \delta_l^i, \qquad g_{il}(\mathcal{P}_n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i, l = 1, 2, ...n \tag{17}$$

II. Linear structure of the Euclidean space:

$$\sigma_{\mathrm{E}}(P, Q) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}(\mathcal{P}_n)\,(x_i(P) - x_i(Q))\,(x_k(P) - x_k(Q)), \qquad \forall P, Q \in \Omega \tag{18}$$

where coordinates $x_i(P)$, $x_i(Q)$, $i = 1, 2, ...n$ of the points P and Q are covariant coordinates of the g-vectors $\mathbf{P}_0\mathbf{P}$, $\mathbf{P}_0\mathbf{Q}$ respectively in the coordinate system K.

III: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues g_k

$$g_k > 0, \qquad k = 1, 2, ..., n \tag{19}$$

IV. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i = 1, 2, ...n \tag{20}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i = 1, 2, ...n$ has always one and only one solution. Conditions I – IV contain a reference to the dimension n of the Euclidean space, which is defined by the relations (15).

Special relations of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ may be not valid for other physical geometries. In some cases these relations may used partly. For instance, the metric dimension may be defined locally. Instead of constraint (15) one uses the condition

$$\forall P_0 \in \delta\Omega, \quad \exists \mathcal{P}_n \equiv \{P_0, P_1, ...P_n\} \subset \delta\Omega, \quad F_n(\mathcal{P}_n) \neq 0, \quad F_k(\mathcal{P}_k) = 0, \quad k > n \tag{21}$$

where $\delta\Omega$ is a infinitesimal region $\delta\Omega \subset \Omega$, and all skeletons $\mathcal{P}_n$ contain only infinitely close points. The conditions (21) determine the metric dimension for locally flat (Riemannian) geometry.

Applications of the discrete geometry to a description of the space-time geometry has been developed since beginning of ninetieth of the twentieth century [5]-[16]. As a result one succeeded to create a unite formalism for description of the continuous space-time geometry and of the discrete one.

Consideration of the physical geometry [2] is important for its application to the space-time geometry, which appears to be a discrete geometry in microcosm. Elementary length λ_0, which is a parameter of the space-time geometry, is connected with the quantum constant $\hbar$. This fact admits one to explain motion of quantum particles as a motion of classical particles in the discrete space-time geometry [16]. This motion appears to be stochastic. It is especially unexpected in application to tachyons. It appears that tachyons may exist, however the tachyon world line wobbles with infinite amplitude. As a result of this wobbling a single tachyon cannot be detected, but the tachyon gas can be detected by its gravitational field. Existence of the tachyon gas may explain the cosmological problem of dark matter [17]. A use of non-Riemannian geometry for description of cosmos admits one to expand the general relativity to a wider class of space-time geometries.

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