Hidden dynamical variables in rotational flow of barotropic fluid

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Abstract

Inviscid barotropic fluid is investigated as a dynamical system by means of variational methods. Conventional description in terms of variables: (density ρ, velocity $\mathbf{v}$, and labelling of stream lines $\boldsymbol{\xi}$) appears to be ineffective for vortical flows, because dynamic equations for $\boldsymbol{\xi}$ contain idefinite parameter. It is possible complete description of barotropic fluid in terms of complex wave function ψ_α, $\alpha = 1, 2$. At returning to description in terms of $\rho, \mathbf{v}$ an additional dynamic variables: spin $\mathbf{s} = (s_1, s_2, s_3)$ appear. Spin describes additional vorticity incorporated into fluid. Appearance of additional variable $\mathbf{s}$ (instead of $\boldsymbol{\xi}$) is essential at description of turbulent phenomena.

Key words: inviscid fluid; hidden variables; wave function; stream lines labelling; hidden vorticity; spin vector; turbulence

1 Introduction

Turbulence phenomena are described usually by the Navier- Stokes equations. The smaler is viscosity μ the greiter turbulence takes place. If μ tends to zero, the Navier- Stokes equations tends to Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\,\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p, \qquad p = p(\rho) = \rho^2 \frac{\partial E}{\partial \rho} \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho \mathbf{v}) = 0 \tag{1.2}$$

where $\rho = \rho(t, \mathbf{x})$ is the fluid density, $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ is the fluid velocity, $p = p(\rho) = \rho^2 \partial E(\rho)/\partial \rho$ is the pressure and $E(\rho)$ is the fluid internal energy per unit mass. Equations (1.1) and (1.2) is written for barotropic fluid. It seems that they do not describe turbulence at all. It is rather strange, because at $\mu = 0$ the turbulence phenomena must be maximal.

There is such a possibility, that there are additional dynamical variables (other than ρ and $\mathbf{v}$), which describe turbulence. Such a possibility is not considered usually, because the system of equations (1.1) and (1.2) is a closed system of dynamical equations. However, nevertheless the system of equations (1.1) and (1.2) is not a complete system of dynamic equations for dynamic system: barotropic fluid S_b. Determination of additional dynamic variables for S_b is a purpose of this publication.

Numerous attempts to obtain dynamic equations (1.1) and (1.2) from a variational principle were failed [1]. However, the variational principle can be obtained, provided one uses the Lin constraint [2]

$$\partial_0 \boldsymbol{\xi} + (\mathbf{v}\boldsymbol{\nabla})\,\boldsymbol{\xi} = 0 \tag{1.3}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_1, \xi_1)$ are variables labelling particles of a fluid. The Lin constraint should be introduced in the action functional as a side condition. It means that the Lin constraint is an additional dynamic equation. However, further investigation shows, that seven equations (1.1), (1.2) and (1.3) form a complete system of dynamic equations for S_b only in the case of a potential flow. In general case additional dynamic equations have the form

$$\partial_0 \xi_\alpha + (\mathbf{v}\boldsymbol{\nabla})\,\xi_\alpha = -\omega \varepsilon_{\alpha\beta\gamma} \Omega^{\beta\gamma}(\boldsymbol{\xi}) \qquad \alpha = 1,2,3 \tag{1.4}$$

where ω is arbitrary quantity,

$$\Omega^{a\mu}(\boldsymbol{\xi}) = \left(\frac{\partial g^\alpha(\boldsymbol{\xi})}{\partial \xi_\mu} - \frac{\partial g^\mu(\boldsymbol{\xi})}{\partial \xi_\alpha} \right) \tag{1.5}$$

and $\mathrm{g}^\alpha(\boldsymbol{\xi})$, $\alpha = 1,2,3$ are functions, which are determined from initial conditions. Equations (1.4) coincide with (1.3), if $\Omega^{a\mu}(\boldsymbol{\xi}) = 0$, i.e. in the case of potential flow.

Equation (1.4) cannot be solved because of indefinite quantity ω.

2 Variational principle

The action functional for Euler equations (1.1), (1.2) has the form

$$\mathcal{A}[\xi, j, p] = \int\limits_{V_x} \left\{ \frac{\mathbf{j}^2}{2\rho_0} - \rho_0 E(\rho) - p_k \left(j^k - \frac{\partial J}{\partial \xi_{0,k}} \right) \right\} d^4 x, \tag{2.1}$$

It contains the side condition (1.3) in the form

$$j^k - \frac{\partial J}{\partial \xi_{0,k}} = 0 \tag{2.2}$$

where $J = J_{\xi/x}$ is the Jacobian determinant

$$J_{\xi/x} = J\left(\xi_{l,k}\right) = \frac{\partial(\xi_0, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} = \det\left|\left|\xi_{l,k}\right|\right|, \tag{2.3}$$

$$\xi_{l,k} \equiv \frac{\partial \xi_l}{\partial x^k} \qquad l,k = 0,1,2,3$$

Dealing with Jacobian J, the following identities are useful

$$\frac{\partial J}{\partial \xi_{i,l}} \xi_{k,l} \equiv J\delta_k^i, \qquad \partial_k \frac{\partial J}{\partial \xi_{0,k}} \equiv 0, \qquad \partial_l \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \equiv 0 \tag{2.4}$$

$$\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \equiv J^{-1} \left(\frac{\partial J}{\partial \xi_{0,k}} \frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}} \frac{\partial J}{\partial \xi_{l,k}} \right) \tag{2.5}$$

Using the first identity (2.4), we obtain

$$\frac{\partial J}{\partial \xi_{0,k}} \xi_{\alpha,k} = j^k \xi_{\alpha,k} = \rho \partial_0 \xi_\alpha + \rho v^\alpha \xi_\alpha = 0 \tag{2.6}$$

It means that (2.6) is equivalent to (1.3).

Variation of (2.1) with respect to ξ_l gives

$$\delta \xi_l : \qquad -\partial_s \left(p_k \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{l,s}} \right) = 0, \qquad l = 0,1,2,3 \tag{2.7}$$

Using (2.4) and (2.5), this equation can be presented in the form

$$\left(\frac{\partial J}{\partial \xi_{0,k}} \frac{\partial J}{\partial \xi_{l,s}} - \frac{\partial J}{\partial \xi_{0,s}} \frac{\partial J}{\partial \xi_{l,k}} \right) p_k = 0 \tag{2.8}$$

Equations (2.8) can be integrated in the form

$$p_k = b \left(\partial_k \varphi + g^\alpha (\boldsymbol{\xi}) \partial_k \xi_\alpha \right), \qquad k = 0,1,2,3 \tag{2.9}$$

where $g^\alpha (\boldsymbol{\xi})$, $\alpha = 1,2,3$ are arbitrary functions of $\boldsymbol{\xi}$, $\varphi = g^0 (\xi_0)$ is a new variable instead of fictitious variable ξ_0, b is a constant. Using identities (2.4), this fact can be tested by a direct substitution of (2.9) in (2.8). Note, that this integration has been produced for incompressed fluid by Clebsch [4, 5] 160 years ago. Let us now substitute p_k from (2.9) into action (2.1). Let us set $b = 1$. We obtain the new action functional

$$\mathcal{A}[\xi, j, \varphi] = \int\limits_{V_x} \left\{ \frac{\mathbf{j}^2}{2\rho_0} - \rho_0 E(\rho_0) - j^k \left(\partial_k \varphi + g^\alpha (\boldsymbol{\xi}) \partial_k \xi_\alpha \right) \right\} d^4 x, \tag{2.10}$$

which is equivalent to action functional(2.1). It contains arbitrary integration functions $g(\boldsymbol{\xi})$. Here

$$j^0 = \rho_0, \qquad \mathbf{j} = \rho_0 \mathbf{v} = \left\{ j^1, j^2, j^3 \right\} \tag{2.11}$$

The integration functions $g(\boldsymbol{\xi})$ are considered as fixed functions of $\boldsymbol{\xi}$.

Variation of (2.10) with respect to ξ_α gives

$$\delta \xi_\alpha : \qquad \rho_0 \Omega^{a\mu} (\boldsymbol{\xi}) \left(\partial_0 \xi_\alpha + (\mathbf{v} \boldsymbol{\nabla}) \xi_\alpha \right) = 0, \tag{2.12}$$

where

$$\Omega^{\alpha\mu}\left(\boldsymbol{\xi}\right)=\left(\frac{\partial g^{\alpha}\left(\boldsymbol{\xi}\right)}{\partial\xi_{\mu}}-\frac{\partial g^{\mu}\left(\boldsymbol{\xi}\right)}{\partial\xi_{\alpha}}\right) \tag{2.13}$$

and $\mathbf{v}$ is determined by the relation

$$\delta j^{\mu}:\qquad v^{\mu}\equiv\frac{j^{\mu}}{\rho_{0}}=\partial_{\mu}\varphi+g^{\alpha}\left(\boldsymbol{\xi}\right)\partial_{\mu}\xi_{\alpha} \tag{2.14}$$

If $\det\left|\left|\Omega^{\alpha\beta}\right|\right|\neq0$, then the Lin constraints

$$\left(\partial_{0}\xi_{\alpha}+\left(\mathbf{v}\boldsymbol{\nabla}\right)\xi_{\alpha}\right)=0 \tag{2.15}$$

follows from (2.12)

However, the matrix $\Omega^{\alpha\beta}$ is antisymmetric and in 3-dimensional space

$$\det\left|\left|\Omega^{\alpha\beta}\right|\right|=\left|\begin{array}{ccc}0 & \Omega^{12} & \Omega^{13}\\ -\Omega^{12} & 0 & \Omega^{23}\\ -\Omega^{13} & -\Omega^{23} & 0\end{array}\right|\equiv0 \tag{2.16}$$

Then it follows from (2.12)

$$\partial_{0}\xi_{\alpha}+\left(\mathbf{v}\boldsymbol{\nabla}\right)\xi_{\alpha}=-\omega\varepsilon_{\alpha\beta\gamma}\Omega^{\beta\gamma}\left(\boldsymbol{\xi}\right)\qquad\alpha=1,2,3 \tag{2.17}$$

where $\omega=\omega\left(t,\boldsymbol{\xi}\right)$ is an arbitrary quantity.

The obtained equation (2.17) contains the initial dynamic equation (1.3) as a special case. For irrotational flow, when $\Omega^{\beta\gamma}\left(\boldsymbol{\xi}\right)=0$, the equation (2.17) turns to (1.3). In the action functional (2.1) the initial relation (1.3) is used as a side constraint. The real hidden variables are more rich, than it is described by the Lin constraint. It is a reason, why the equation (2.17) is not obtained from the action functional (2.1).

If $\omega\left(t,\boldsymbol{\xi}\right)\neq0$, the dynamic equations (2.17) describe a violation of the Lin constraints (1.3). One obtains another labelling of the stream lines, than that one, which is described by the Lin constraints (1.3). If the flow is irrotational, and $\boldsymbol{\Omega}^{\alpha\beta}=0$, the labelling does not depend on the arbitrary quantity $\omega\left(t,\boldsymbol{\xi}\right)$.

Thus, although solution $\rho_{0},\mathbf{v}$ of the Cauchy problem for the Euler system of hydrodynamic equation (1.1), (1.2) is unique (in the sense, that it does not contain indefinite quantities), the solution $\rho_{0},\mathbf{v},\boldsymbol{\xi}$ for the Cauchy problem of the complete system of hydrodynamic equations (1.1), (1.2), (2.17) is not unique (in the sense, that it contains indefinite quantity $\omega\left(t,\mathbf{x}\right)$). The reason of this nonuniqueness is influence of interfusion. Our consideration is formal. One cannot understand mechanism of the interfusion influence from this consideration. Nevertheless this influence takes place, and it should be investigated more closely.

Remark. Variables $\boldsymbol{\xi}=\left(\xi_{1},\xi_{2},\xi_{3}\right)$ are not determined uniquely. It may be a result of abortive choice of dynamic variables $\boldsymbol{\xi}$.

3 Description in terms of the complex fluid potential

Let us introduce new complex dynamic variables $\psi = (\psi_1, \psi_2)$ determining them by relations

$$j^0 = \rho = \sum_{\alpha=1,2} \psi^*_\alpha \psi_\alpha \equiv \psi^* \psi \tag{3.1}$$

$$v^\beta (t, \mathbf{x}) = -\frac{i}{2} \sum_{\alpha=1}^{2} \frac{\psi^*_\alpha \frac{\partial \psi_\alpha}{\partial \xi_\beta} - \frac{\partial \psi^*_\alpha}{\partial \xi_\beta} \psi_\alpha}{\psi^* \psi} \equiv -\frac{i}{2} \frac{\psi^* \partial_\beta \psi - \partial_\beta \psi^* \psi}{\psi^* \psi}, \quad \beta = 1, 2, 3 \tag{3.2}$$

This are such variables, where it is possible complete description of dynamic system, described by the action (2.10). These variables are known as the complex fluid potential ψ (wave function [3]). It is used in quantum mechanics as the wave function.

The complex fluid potential is defined via variables $\rho, \mathbf{v}$ by differential relations, But ψ cannot be expressed via $\rho, \mathbf{v}$ solely. In terms of ψ the action functional (2.10) takes the form

$$\begin{aligned}\mathcal{A} [\psi, \psi^*] = & \int\limits_{V_x} \left\{ \frac{i}{2} \sum_{\alpha=1}^{\alpha=2} (\psi^*_\alpha \partial_0 \psi_\alpha - \partial_0 \psi^*_\alpha \cdot \psi_\alpha) - \rho E (\rho) \right\} d^4 x \\ & + \int\limits_{V_x} \frac{\mathbf{1}}{8} \left(\sum_{\alpha=1}^{\alpha=2} (\psi^*_\alpha \partial_\beta \psi_\alpha - \partial_\beta \psi^*_\alpha \cdot \psi_\alpha) \right)^2 d^4 x \\ & - \int\limits_{V_x} \frac{1}{4} \sum_{\gamma=1}^{\gamma=2} \sum_{\alpha \neq \gamma=1}^{\alpha \neq \gamma=2} \left((\psi^*_\alpha \partial_\beta \psi_\alpha - \partial_\beta \psi^*_\alpha \cdot \psi_\alpha) \left(\psi^*_\gamma \partial_\beta \psi_\gamma - \partial_\beta \psi^*_\gamma \cdot \psi_\gamma \right) \right) d^4 x\end{aligned} \tag{3.3}$$

where

$$\rho = \psi \psi^* = \sum_{\alpha=1}^{\alpha=2} \psi_\alpha \psi^*_\alpha \tag{3.4}$$

Variation of (3.3) with respect to ψ^*_α $\alpha = 1, 2$ leads to dynamic equations

$$\begin{aligned} & i \partial_0 \psi_\alpha + \frac{1}{2} ((\psi^*_\alpha \partial_\beta \psi_\alpha - \partial_\beta \psi^*_\alpha \cdot \psi_\alpha)) \partial_\beta \psi_\alpha - \frac{\partial}{\partial \rho} (\rho E (\rho)) \psi_\alpha \\ & + \frac{1}{2} \partial_\beta \left(\psi^*_{3-\alpha} \partial_\beta \psi_{3-\alpha} - \partial_\beta \psi^*_{3-\alpha} \cdot \psi_{3-\alpha} \right) \psi_\alpha = 0, \quad \alpha = 1, 2 \end{aligned} \tag{3.5}$$

Variation of (3.3) with respect to ψ_α leads to dynamic equations which are complex conjugated to (3.5).

The complex fluid potential $\psi = \{\psi_\alpha\}, \alpha = 1, 2$ is a 2-component complex function. It is constructed from Clebsch potentials (2.9).

Dynamic equations in terms of ψ do not contain indefinite quantities of the type ω in (2.17). However, indefiniteness in the definitions of variables $\boldsymbol{\xi}$ remains, because variables ρ,$\mathbf{v}$, $\boldsymbol{\xi}$ are determined via ψ from relations (3.1) and (3.2)

The complex fluid potential ψ is a natural attribute of fluid dynamics, but it was not known to researchers dealing with fluid dynamics. However, it was used in quantum mechanics under name of wave function. Nature of the complex fluid potential ψ was not known during the whole XX century. In quantum mechanics the wave function ψ was considered as an axiomatical object of unknown nature. This fact has led to numerous interpretations of quantum mechanics which were generated by unknown meaning of the wave function ψ.

The same action (2.10) written in in terms of the fluid potential ψ generates dynamic equations (3.5). These equations are resolved with respect to time derivatives and do not contain indefinite quantities. But now the indefiniteness is placed in definition of the fluid potential ψ via variables $\rho, \mathbf{v}, \boldsymbol{\xi}$. If initial conditions are given for variables $\rho, \mathbf{v}, \boldsymbol{\xi}$ in the form

$$\rho\left(0,\mathbf{x}\right)=\rho_{\mathrm{in}}\left(\mathbf{x}\right), \quad v^{\alpha}\left(0,\mathbf{x}\right)=g^{\alpha}\left(\mathbf{x}\right), \quad \xi_{\alpha}\left(0,\mathbf{x}\right)=x^{\alpha} \tag{3.6}$$

dynamic equations, written in terms of $\rho, \mathbf{v}, \boldsymbol{\xi}$, cannot be solved uniquely, because of indefinite quantity $\omega\left(t,\mathbf{x}\right)$ in (2.17). Dynamic equations (2.17) cannot be solved uniquely at the initial conditions (3.6), because in this case the initial conditions $\psi_{\alpha}\left(0,\mathbf{x}\right)$ cannot be determined uniquely via variables $\rho, \mathbf{v}, \boldsymbol{\xi}$. However, if initial conditions $\psi_{\alpha}\left(0,\mathbf{x}\right)$ for fluid potential ψ are given, the equation (3.6) can be solved, because they do not contain indefinite parameters.

It means that the dynamic variables $\rho_0, \mathbf{v}, \boldsymbol{\xi}$ do not form a complete set of dynamic variables of a barotropic fluid, whereas components ψ_{α} of the fluid potential ψ form a complete set of dynamic variables. In the case of irrotational flow the dynamic variables $\rho_0, \mathbf{v}, \boldsymbol{\xi}$ form a complete set of dynamic variables of a barotropic fluid. In the case of vortical flow the dynamic variables $\rho_0, \mathbf{v}, \boldsymbol{\xi}$ do not form a complete set of dynamic variables, which are needed for description of a vortical flow of a barotropic fluid. Thus, additional dynamic (hidden) variables ψ_{α}, which are needed for complete description of a barotropic fluid, are connected with rotational motion of a fluid.

It s possible to use real hidden variables $\mathbf{s}=\{s_1,s_2,s_3\}$ defined by relations

$$s_1=\frac{\psi^*\sigma_1\psi}{\rho}=\frac{1}{\rho}\psi^*\left(\begin{array}{cc}0 & -i\\ i & 0\end{array}\right)\psi=\frac{1}{\rho}\left(-i\psi_1^*\psi_2+i\psi_2^*\psi_1\right) \tag{3.7}$$

$$s_2=\frac{\psi^*\sigma_2\psi}{\rho}=\frac{1}{\rho}\psi^*\left(\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right)\psi=\frac{1}{\rho}\left(\psi_1^*\psi_2+\psi_2^*\psi_1\right) \tag{3.8}$$

$$s_3=\frac{\psi^*\sigma_3\psi}{\rho}=\frac{1}{\rho}\psi^*\left(\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right)\psi=\frac{1}{\rho}\left(\psi_1^*\psi_1-\psi_2^*\psi_2\right) \tag{3.9}$$

Dynamic equations for $\mathbf{s}$ are obtained by means of (3.5). In quantum mechanics variables $\mathbf{s}$ are known as spin variables, describing angular momentum of particles.

4 Concluding remarks

We have considered the barotropic fluid. However, the wave function appears as a result of integration of equation (2.7), which does not depend on the fluid state. Thus, the obtained result is valid for any inviscid fluid. The viscid barotropic fluid is described by the Navier-Stokes equations. If viscosity tends to zero, the Navier-Stokes equations tends to the Euler equations. If the viscosity reduces, the turbulent phenomena increase. If the viscosity vanishes, the turbulence becomes maximal. At these conditions it seems rather natural, that the conventional description by means of Euler equations cannot describe irregular stream lines, which are connected with possible turbulence. Maybe, a use of the fluid description in terms of fluid potential (wave function) would be useful for description of turbulent phenomena. Description in terms of ψ admits one to follow an evolution of stream lines, although such a description is not connected with initial values of observable variables $\rho, \mathbf{v}, \boldsymbol{\xi}$. Besides, it is not clear, how can one introduce the fluid potential into description of viscid fluid, which is not described by a variational principle.

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