# Different representations of Euclidean geometry and their application to the space-time geometry

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**Abstract**

Three different representation of the proper Euclidean geometry are considered. They differ in the number of basic elements, from which the geometrical objects are constructed. In E-representation there are three basic elements (point, segment, angle) and no additional structures. V-representation contains two basic elements (point, vector) and additional structure: linear vector space. In $\sigma$-representation there is only one basic element (point) and additional structure: world function $\sigma = \rho^2/2$, where $\rho$ is the distance. The concept of distance appears in all representations. However, as a structure, determining the geometry, the distance appears only in the $\sigma$-representation. The $\sigma$-representation is most appropriate for modification of the proper Euclidean geometry. Practically any modification of the proper Euclidean geometry turns it into multivariant geometry, where there are many vectors $\mathbf{Q}_0\mathbf{Q}_1, \mathbf{Q}_0\mathbf{Q}_1', ...$, which are equal to the vector $\mathbf{P}_0\mathbf{P}_1$, but they are not equal between themselves, generally speaking. Concept of multivariance is very important in application to the space-time geometry. The real space-time geometry is multivariant. Multivariance of the space-time geometry is responsible for quantum effects.

*Key words: general geometric relations; special Euclidean relation; monistic conception of geometry*

## 1 Introduction

The proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ was constructed at the supposition, that the geometry is continuous and unlimitedly divisible. It means that all geometrical objects in $\mathcal{G}_{\mathrm{E}}$ can be constructed of blocks. Such geometrical objects as segments of the straight line and angles can be used as constructive blocks. Construction of $\mathcal{G}_{\mathrm{E}}$ in the form of a logical construction is connected with the possibility that any geometrical object can be constructed of blocks. If the space-time geometry appears to be discrete or restrictedly divisible, it can not be presented as a logical construction. Any restrictedly divisible geometry is to be constructed as a result of deformation of $\mathcal{G}_{\mathrm{E}}$. Deformation of a unlimitedly divisible geometry does not conserve the property of unlimitedly divisibility. To realize deformation of $\mathcal{G}_{\mathrm{E}}$, it must be presented in the form of a monistic conception, where the Euclidean metric $\rho_{\mathrm{E}}$ is a unique fundamental quantity describing $\mathcal{G}_{\mathrm{E}}$. All geometric objects and all geometric relations are to be expressed in terms and only in terms of Euclidean metric. Replacing Euclidean metric $\rho_{\mathrm{E}}$ by another metric $\rho$, one realizes deformation of $\mathcal{G}_{\mathrm{E}}$. As a result one obtains some generalized geometry $\mathcal{G}$, described by the metric $\rho$. It is more convenient to use world function $\sigma = \frac{1}{2}\rho^2$ instead of metric $\rho$, because world function is always real, whereas metric is imaginary for spacelike intervals.

An attempt of the $\mathcal{G}_{\mathrm{E}}$ metric description in terms of metric was made in so called distance geometry [1]. Unfortunately, Blumental failed to express the straight line in terms of a metric. As a result the conception of distance geometry appeared to be not monistic, and one cannot deform the distance geometry.

The proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ studies mutual dispositions of geometrical objects (figures) in the space (in the set $\Omega$ of points). Any geometrical object $\mathcal{O}$ is a subset $\mathcal{O} \subset\Omega$ of points. Relations between different objects are relations of equivalence, when two different objects $\mathcal{O}_1$ and $\mathcal{O}_2$ are considered to be equivalent ($\mathcal{O}_1 \mathrm{eqv} \mathcal{O}_2$). The geometrical object $\mathcal{O}_1$ is considered to be equivalent to the geometrical object $\mathcal{O}_2$, if after corresponding displacement the geometrical object $\mathcal{O}_1$ coincides with the geometrical object $\mathcal{O}_2$.

The geometrical object is considered to be constructed of basic elements (blocks). There are at least three representations of Euclidean geometry, which differ in the number and in the choice of basic elements (primary concepts). It should take into account, that these representation has nothing to do with the problem of the Euclidean geometry construction. These representations are used, when $\mathcal{G}_{\mathrm{E}}$ has been already constructed.

The first representation (Euclidean representation, or E-representation) of the Euclidean geometry was obtained by Euclid many years ago. The basic elements in the E-representation are point, segment and angle. The segment is a segment of the straight line. It consists of infinite number of points. The segment is determined uniquely by its end points. The angle is a figure, formed by two segments provided the end of one segment coincides with the end of other one. Properties of basic elements are described by a system of axioms. Any geometrical object may be considered to be some composition of blocks (*point, segment, angle*). The number of the geometric object constituents may be infinite. The segments determine distances. The angles determine the mutual orientation of segments. Comparison of different geometrical objects (figures) $\mathcal{O}_1$ and $\mathcal{O}_2$ is produced by their displacement and superposition. If two figures coincide at superposition, they are considered to be equivalent (equal). The process of displacement in itself is not formalized in E-representation. However, the law of the geometric objects displacement is used at the formulation of the Euclidean geometry propositions and at their proofs.

The second representation (vector representation, or V-representation) of the Euclidean geometry contains two basic elements (*point, vector*). From the viewpoint of E-representation the vector is a directed segment of straight, determined by two points. One point is the origin of the vector, another point is the end of the vector. But such a definition of vector takes place only in E-representation, where the vector is a secondary (derivative) concept. In V-representation the vector is defined axiomatically as an element of a linear vector space, where two operations under vectors are defined. These operations (addition of two vectors and multiplication of a vector by a real number) formalize the law of displacement of vectors. Strictly, these operations are simply some formal operations in the linear vector space. They begin to describe the law of displacement only after introduction of the scalar product of vectors in the vector space. After introduction of the scalar product the linear vector space becomes to be the Euclidean space, and the abstract vector may be considered as a directed segment of straight, determined by two points. However, it is only interpretation of a vector in the Euclidean representation. In the V-representation the vector is a primary object. It is an element of the linear vector space. Nevertheless interpretation of a vector as a directed segment of straight is very important at construction of geometrical objects (figures) from points and vectors.

The E-representation contains three basic elements: *point, segment* and *angle*. The V-representation contains only two basic elements: *point* and *vector*. The angle of the E-representation is replaced by the linear vector space, which is a structure, describing interrelation of two vectors (directed segments). The vector has some properties as an element of the linear vector space. Any geometrical figure may be constructed of points and vectors. It means that the method of construction of any figure may be described in terms of points and vectors. The properties of a vector as an element of the linear vector space admit one to describe properties of displacement of figures and their compositions.

In V-representation the angle appears to be an derivative element. It is determined by two vectors (by their lengths and by the scalar product between these vectors). In the V-representation the angle $\theta$ between two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$ is defined by the relation

$$|\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{P}_0\mathbf{P}_2| \cos\theta = (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2) \tag{1.1}$$

where $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2)$ is the scalar product of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$. The quantities $|\mathbf{P}_0\mathbf{P}_1|$ and $|\mathbf{P}_0\mathbf{P}_2|$ are their lengths

$$|\mathbf{P}_0\mathbf{P}_1|^2 = (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1), \qquad |\mathbf{P}_0\mathbf{P}_2|^2 = (\mathbf{P}_0\mathbf{P}_2.\mathbf{P}_0\mathbf{P}_2) \tag{1.2}$$

Thus, transition from the E-representation with three basic elements to the V-representation with two basic elements is possible, provided the properties of one basic element (vector) are determined by the fact, that the vector is an element of the linear vector space with the scalar product, given on it.

Is it possible a further reduction of the number of basic elements in the representation of the Euclidean geometry? Yes, it is possible. The representation (in terms world function, or $\sigma$-representation) of the Euclidean geometry may contain only one basic element (point), provided there are some constraints, imposed on any two points of the Euclidean space.

The transition from the E-representation to the V-representation reduces the number of basic elements. Simultaneously this transition generates a new structure (linear vector space), which determine properties of new basic element (vector).

The transition from the V-representation to $\sigma$-representation also reduces the number of basic elements. Only one basic element (*point*) remains. Simultaneously this transition replaces the linear vector space by a new two-point structure (*world function*), which describes interrelation of two points instead of two vectors. The world function $\sigma$ is defined by the relation

$$\sigma: \qquad \Omega\times\Omega\rightarrow\mathbb{R}, \qquad \sigma\left(P,Q\right)=\sigma\left(Q,P\right), \qquad \sigma\left(P,P\right)=0, \qquad \forall P,Q\in\Omega \tag{1.3}$$

as a single-valued function of two points. The world function $\sigma$ is connected with the distance $\rho$ by the relation

$$\sigma\left(P,Q\right)=\frac{1}{2}\rho^{2}\left(P,Q\right), \qquad \forall P,Q\in\Omega \tag{1.4}$$

The world function of the proper Euclidean space is constrained by a series of restrictions (see below).

There is a difference between the structure of the V-representation and that of $\sigma$-representation. Linear vector space as a structure of the V-representation reflects the symmetry properties of the Euclidean geometry. These symmetries of the Euclidean space admit the motion group of the Euclidean space (translations, rotations). These motion groups admit one to move blocks without deformations and to construct geometric objects from blocks. The motion groups admit one also to compare different geometrical objects, moving them in the space.

The world function $\sigma_{\mathrm{E}}$ as a structure of the proper Euclidean space reflects all properties of the proper Euclidean space, but not only its symmetries. The world function $\sigma_{\mathrm{E}}$ describes the properties of the proper Euclidean geometry completely. Any change of the world function $\sigma_{\mathrm{E}}$ is a deformation of the proper Euclidean space, which changes its properties.

It should note in this connection, that there are different points of view on that, what is a geometry. Well known mathematician Felix Klein supposed that symmetries of a geometry are the most essential properties of the geometry, and there exist no geometries without a symmetry. For instance, he wrote that the Riemannian geometry is not a geometry. It is rather a geography or topography. Such a viewpoint is characteristic for mathematicians. Most of them believe, that a geometry is a logical construction, and there exist no nonaxiomatizable geometries.

Alternative viewpoint is characteristic for physicists, who believe, that a geometry is a science on a shape of geometrical objects and on their mutual disposition.

At such an approach it is of no importance, whether or not the geometry has any symmetries and whether or not it is axiomatizable. If a geometry is science on the disposition of geometrical objects, the geometry is described completely by the world function $\sigma$. Approach of physicists seems to be more realistic, because the matter distribution influences on the space-time geometry, and one cannot be sure, that the space-time geometry is uniform, and it has some symmetries.

Existence of the $\sigma$-representation, containing only one basic element, means that all geometrical objects and all relations between them may be recalculated to the $\sigma$-representation, i.e. expressed in terms of the world function and only in terms of the world function.

If we want to construct a generalized geometry, we are to modify properties of the proper Euclidean geometry. It means that we are to modify properties of basic elements of the proper Euclidean geometry. In the E-representation we have three basic elements (*point, segment, angle*). Their properties are connected, because finally the segment and the angle are simply sets of points. Modification the three basic elements cannot be independent. It is very difficult to preserve connection between the modified basic elements of a generalized geometry. Nobody does modify the proper Euclidean geometry in the E-representation.

In V-representation there are two basic elements of geometry (*point, vector*), and in some cases the modification of the proper Euclidean geometry is possible. However, the V-representation contains such a structure, as the linear vector space. One cannot avoid this structure, because it is not clear, what structure may be used instead. Modification of the proper Euclidean geometry in the framework of the linear vector space is restricted rather strong. (It leads to the pseudo-Euclidean geometries and to the Riemannian geometries). Besides, it appears, that there exist such modifications of the proper Euclidean geometry, which are incompatible with the statement that any geometrical object may be constructed of points and vectors.

The $\sigma$-representation of the proper Euclidean geometry is most appropriate for modification, because it contains only one basic element (*point*). Any modification of the proper Euclidean geometry is accompanied by a modification of the structure, associated with the $\sigma$-representation. This structure is distance (world function), and any modification of the proper Euclidean geometry is accompanied by a modification of world function (distance), and vice versa. The meaning of distance is quite clear. This concept appears in all representations of the proper Euclidean geometry, but only in the $\sigma$-representation the distance (world function) plays the role of a structure, determining the geometry. Modification of the distance means a deformation of the proper Euclidean geometry.

The V-representation appeared in the nineteenth century, and most of contemporary mathematicians and physicists use this representation. The $\sigma$-representation appeared recently in the end of the twentieth century. Besides, it appears in implicit form in the papers, devoted to construction of T-geometry [2, 3]. In these papers the term "$\sigma$-representation" was not mentioned, and the $\sigma$-representation was considered as an evident possibility of the geometry description in terms of the world function. Apparently, such a possibility was evident only for the author of the papers, but not for readers. Now we try to correct our default and to discuss properties of the $\sigma$-representation of the proper Euclidean geometry.

# 2 $\sigma$-representation of the proper Euclidean geometry

Let $\Omega$ be the set of points of the Euclidean space. The distance $\rho = \rho(P_0, P_1)$ between two points $P_0$ and $P_1$ of the Euclidean space is known as the Euclidean metric. In E-representation as well as in V-representation we have

$$\rho^2(P_0, P_1) = |\mathbf{P}_0\mathbf{P}_1|^2 = (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1), \qquad \forall P_0, P_1 \in \Omega \tag{2.1}$$

It is convenient to use the world function $\sigma(P_0, P_1) = \frac{1}{2}\rho^2(P_0, P_1)$ as a main characteristic of the Euclidean geometry. To approach this, one needs to describe properties of any vector $\mathbf{P}_0\mathbf{P}_1$ as an element of the linear vector space in terms of the world function $\sigma(P_0, P_1)$, which is associated with the vector $\mathbf{P}_0\mathbf{P}_1$. In $\sigma$-representation the vector $\mathbf{P}_0\mathbf{P}_1$ is defined as the ordered set of two points $\mathbf{P}_0\mathbf{P}_1 = \{P_0, P_1\}$.

We can to add two vectors, when the end of one vector is the origin of the other one

$$\mathbf{P}_0\mathbf{Q}_1 = \mathbf{P}_0\mathbf{Q}_0 + \mathbf{Q}_0\mathbf{Q}_1 \quad \text{or} \quad \mathbf{Q}_0\mathbf{Q}_1 = \mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{Q}_0, \tag{2.2}$$

Substituting the point $Q_0$ by $P_1$, one obtains

$$\mathbf{P}_1\mathbf{Q}_1 = \mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1 \tag{2.3}$$

Then according to properties of the scalar product in the Euclidean space we obtain from the second relation (2.2)

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) - (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_0) \tag{2.4}$$

Besides, according to the properties of the scalar product in the Euclidean geometry we have

$$|\mathbf{P}_1\mathbf{Q}_1|^2 = (\mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1) = |\mathbf{P}_0\mathbf{Q}_1|^2 - 2(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) + |\mathbf{P}_0\mathbf{P}_1|^2 \tag{2.5}$$

Using definition of the world function

$$\sigma(P_0, P_1) = \sigma(P_1, P_0) = \frac{1}{2}|\mathbf{P}_0\mathbf{P}_1|^2, \qquad \forall P_0, P_1 \in \Omega \tag{2.6}$$

we obtain from (2.5) and (2.6)

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) = \sigma(P_0, P_1) + \sigma(P_0, Q_1) - \sigma(P_1, Q_1), \qquad \forall P_0, P_1, Q_1 \in \Omega \tag{2.7}$$

Using (2.7) one obtains from (2.4), that for any two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ the scalar product has the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma(P_0, Q_1) + \sigma(P_1, Q_0) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1), \qquad \forall P_0, P_1, Q_0, Q_1 \in \Omega \tag{2.8}$$

Setting $Q_0 = P_0$ in (2.8) and comparing with (2.7), we obtain

$$\sigma(P_0, P_0) = \frac{1}{2}|\mathbf{P}_0\mathbf{P}_0|^2 = 0, \qquad \forall P_0 \in \Omega \tag{2.9}$$

$n$ vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ... \ \mathbf{P}_0\mathbf{P}_n$ are linear dependent, if and only if the Gram's determinant $F_n(\mathcal{P}^n)$, $\mathcal{P}^n \equiv \{P_0, P_1, ...P_n\}$ vanishes

$$F_n(\mathcal{P}^n) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)|| = 0, \qquad i, k = 1, 2, ...n \tag{2.10}$$

In the $\sigma$-representation the condition (2.10) is written in the developed form

$$F_n(\mathcal{P}^n) \equiv \det ||\sigma(P_0, P_i) + \sigma(P_0, P_k) - \sigma(P_i, P_k)|| = 0, \qquad i, k = 1, 2, ...n \tag{2.11}$$

If $\sigma$ is the world function of $n$-dimensional Euclidean space, it satisfies the following relations.

I. Definition of the dimension and introduction of the rectilinear coordinate system:

$$\exists \mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_k\left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{2.12}$$

where $F_n(\mathcal{P}^n)$ is the Gram's determinant (2.11). Vectors $P_0P_i$, $i = 1, 2, ...n$ are basic vectors of the rectilinear coordinate system $K_n$ with the origin at the point $P_0$. The covariant metric tensor $g_{ik}(\mathcal{P}^n)$, $i, k = 1, 2, ...n$ and the contravariant one $g^{ik}(\mathcal{P}^n)$, $i, k = 1, 2, ...n$ in $K_n$ are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}(\mathcal{P}^n) g_{lk}(\mathcal{P}^n) = \delta_l^i, \qquad g_{il}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i, l = 1, 2, ...n \tag{2.13}$$

$$F_n(\mathcal{P}^n) = \det ||g_{ik}(\mathcal{P}^n)|| \neq 0, \qquad i, k = 1, 2, ...n \tag{2.14}$$

II. Linear structure of the Euclidean space:

$$\sigma(P, Q) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}(\mathcal{P}^n)(x_i(P) - x_i(Q))(x_k(P) - x_k(Q)), \qquad \forall P, Q \in \Omega \tag{2.15}$$

where coordinates $x_i(P)$, $i = 1, 2, ...n$ of the point $P$ are covariant coordinates of the vector $\mathbf{P}_0\mathbf{P}$, defined by the relation

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i = 1, 2, ...n \tag{2.16}$$

III: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues

$$g_k > 0, \qquad k = 1, 2, ..., n \tag{2.17}$$

IV. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i = 1, 2, ...n \tag{2.18}$$

considered to be equations for determination of the point $P$ as a function of coordinates $y = \{y_i\}$, $i = 1, 2, ...n$ has always one and only one solution. Conditions I $\div$ IV contain a reference to the dimension $n$ of the Euclidean space.

One can show that conditions I $\div$ IV are the necessary and sufficient conditions of the fact that the set $\Omega$ together with the world function $\sigma$, given on $\Omega$, describes the $n$-dimensional Euclidean space [2].

Thus, in the $\sigma$-representation the Euclidean geometry contains only one primary geometrical object: the point. Any two points are described by the world function $\sigma$, which satisfies conditions I $\div$ IV. Any geometrical figure and any relation can be described in terms of the world function and only in terms of the world function.

In the $\sigma$-representation the vector $\mathbf{P}_0\mathbf{P}_1$ is a ordered set $\{P_0, P_1\}$ of two points. Scalar product of two vectors is defined by the relation (2.8).

Two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are collinear (linear dependent), if

$$\mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2 = |\mathbf{P}_0\mathbf{P}_1|^2 \, |\mathbf{Q}_0\mathbf{Q}_1|^2 \tag{2.19}$$

Two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are equivalent (equal), if

$$\mathbf{P}_0\mathbf{P}_1 = \mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \wedge |\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \tag{2.20}$$

Vector $\mathbf{S}_0\mathbf{S}_1$ with the origin at the given point $S_0$ is the sum of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$,

$$\mathbf{S}_0\mathbf{S}_1 = \mathbf{S}_0\mathbf{R} + \mathbf{R}\mathbf{S}_1, \tag{2.21}$$

if the points $S_1$ and $R$ satisfy the relations

$$\mathbf{S}_0\mathbf{R} = \mathbf{P}_0\mathbf{P}_1, \qquad \mathbf{R}\mathbf{S}_1 = \mathbf{Q}_0\mathbf{Q}_1 \tag{2.22}$$

In the developed form it means that the points $S_1$ and $R$ satisfy the relations

$$(\mathbf{S}_0\mathbf{R}.\mathbf{P}_0\mathbf{P}_1) = |\mathbf{S}_0\mathbf{R}| \cdot |\mathbf{P}_0\mathbf{P}_1| , \qquad |\mathbf{S}_0\mathbf{R}| = |\mathbf{P}_0\mathbf{P}_1| \tag{2.23}$$

$$(\mathbf{R}\mathbf{S}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{R}\mathbf{S}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| , \qquad |\mathbf{R}\mathbf{S}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \tag{2.24}$$

where scalar products are expressed via corresponding world functions by the relation (2.8). The points $P_0, P_1, Q_0, Q_1, S_0$ are given. One can determine the point $R$ from two equations (2.23). As far as the world function satisfy the conditions I $\div$ IV, the geometry is the Euclidean one, and the equations (2.23) have one and only one solution for the point $R$. When the point $R$ has been determined, one can determine the point $S_1$, solving two equations (2.24). They also have one and only one solution for the point $S_1$.

Result of summation does not depend on the choice of the origin $S_0$ in the following sense. Let the sum of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ be defined with the origin at the point $S'_0$ by means of the conditions

$$\mathbf{S}'_0\mathbf{S}'_1 = \mathbf{S}'_0\mathbf{R}' + \mathbf{R}'\mathbf{S}'_1 \tag{2.25}$$

where the points $S'_1$ and $R'$ satisfy the relations

$$\mathbf{S}'_0\mathbf{R}' = \mathbf{P}_0\mathbf{P}_1, \qquad \mathbf{R}'\mathbf{S}'_1 = \mathbf{Q}_0\mathbf{Q}_1 \tag{2.26}$$

Then in force of conditions I $\div$ IV the geometry is the Euclidean one, and there is one and only one solution of equations (2.26) and

$$\mathbf{S}_0\mathbf{S}_1 = \mathbf{S}'_0\mathbf{S}'_1 \tag{2.27}$$

in the sense, that

$$\mathbf{S}_0\mathbf{S}_1 = \mathbf{S}'_0\mathbf{S}'_1 : \qquad (\mathbf{S}_0\mathbf{S}_1.\mathbf{S}'_0\mathbf{S}'_1) = |\mathbf{S}_0\mathbf{S}_1| \cdot |\mathbf{S}'_0\mathbf{S}'_1| \wedge |\mathbf{S}_0\mathbf{S}_1| = |\mathbf{S}'_0\mathbf{S}'_1| \tag{2.28}$$

Let us stress that in the $\sigma$-representation the sum of two vectors does not depend on the choice of the origin of the resulting vector, because the world function satisfies the Euclideaness conditions I $\div$ IV. If the world function does not satisfy the conditions I $\div$ IV, the result of summation may depend on the origin $S_0$, as well as on the order of vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{Q}_0\mathbf{Q}_1$ at summation. Besides, the result may be multivariant even for a fixed point $S_0$, because the solution of equations (2.22) may be not unique.

Multiplication of the vector $\mathbf{P}_0\mathbf{P}_1$ by a real number $\alpha$ gives the vector $\mathbf{S}_0\mathbf{S}_1$ with the origin at the point $S_0$

$$\mathbf{S}_0\mathbf{S}_1 = \alpha\mathbf{P}_0\mathbf{P}_1 \tag{2.29}$$

Here the points $P_0, P_1, S_0$ are given, and the point $S_1$ is determined by the relations

$$(\mathbf{S}_0\mathbf{S}_1.\mathbf{P}_0\mathbf{P}_1) = \operatorname{sgn}(\alpha) |\mathbf{S}_0\mathbf{S}_1| \cdot |\mathbf{P}_0\mathbf{P}_1|, \qquad |\mathbf{S}_0\mathbf{S}_1| = |\alpha| \cdot |\mathbf{P}_0\mathbf{P}_1| \tag{2.30}$$

Because of conditions I $\div$ IV there is one and only one solution of equations (2.30) and the solution does not depend on the point $S_0$.

# 3 Uniqueness of operations in the proper Euclidean geometry

To define operations under vectors (equality, summation, multiplication) in the $\sigma$-representation, one needs to solve algebraic equations (2.20), (2.26), (2.29), which are reduced finally to equations (2.20), defining the equality operation.

Equality of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$, ($\mathbf{P}_0\mathbf{P}_1 = \mathbf{Q}_0\mathbf{Q}_1$) is defined by two algebraic equations

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1|, \qquad |\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \tag{3.1}$$

The number of equations does not depend on the dimension of the proper Euclidean space.

In V-representation the number of equations, determining the equality of vectors, is equal to the dimension of the Euclidean space. To define equality of $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$, one introduces the rectilinear coordinate system $K_n$ with basic vectors $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{n-1}$ and the origin at the point $O$. Covariant coordinates $x_k = (\mathbf{P}_0\mathbf{P}_1)_k$ and $y_k = (\mathbf{Q}_0\mathbf{Q}_1)_k$ are defined by the relations

$$x_k \equiv (\mathbf{P}_0\mathbf{P}_1)_k = (\mathbf{e}_k.\mathbf{P}_0\mathbf{P}_1), \qquad y_k \equiv (\mathbf{Q}_0\mathbf{Q}_1)_k = (\mathbf{e}_k.\mathbf{Q}_0\mathbf{Q}_1) \qquad k = 0, 1, ...n-1 \tag{3.2}$$

The equality equations of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ in V-representation have the form

$$x_k = y_k, \qquad k = 0, 1, ..., n-1 \tag{3.3}$$

According to the linear structure of the Euclidean space (2.15) and due to definition of the scalar product (2.8) we obtain

$$|\mathbf{P}_0\mathbf{P}_1|^2 = g^{ik}x_i x_k, \qquad |\mathbf{Q}_0\mathbf{Q}_1|^2 = g^{ik}y_i y_k, \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = g^{ik}x_i y_k \tag{3.4}$$

where

$$g^{ik}g_{lk} = \delta^i_l, \qquad g_{il} = (\mathbf{e}_i.\mathbf{e}_l), \qquad i, k = 0, 1, ..., n-1 \tag{3.5}$$

A summation $0 \div (n-1)$ is produced on repeated indices.

Due to relations (3.4) two equality relations (3.1) take the form

$$g^{kl}x_k y_l = g^{kl}x_k x_l, \qquad g^{kl}x_k x_l = g^{kl}y_k y_l \tag{3.6}$$

By means of the second equation (3.6) the first equation (3.6) may be written in the form

$$g^{kl}(x_k - y_k)(x_l - y_l) = 0 \tag{3.7}$$

According to the III condition (2.17) the matrix of the metric tensor $g^{kl}$ has only positive eigenvalues. In this case the equation (3.7) has only trivial solution for $x_k - y_k$, $k = 0, 1, ...n-1$. Then the equation (3.7) is equivalent to $n$ equations

$$x_l = y_l, \qquad l = 0, 1, ..., n-1 \tag{3.8}$$

Thus, in the $\sigma$-representation two equations (3.1) of the vector equality are equivalent to $n$ equations (3.3) of vector equality in V-representation. This equivalency is conditioned by the linear structure (2.15) of the Euclidean space and by the positive distinctness of the Euclidean metric. In the pseudo-Euclidean space the matrix of the metric tensor has eigenvalues of different sign. In this case the relations (3.7) and (3.8) cease to be equivalent.

# 4 Generalization of the proper Euclidean geometry

Let us consider a simple example of the proper Euclidean geometry modification. The matrix $g^{ik}$ of the metric tensor in the rectilinear coordinate system $K_n$ with basic vectors $\mathbf{e}_0, \mathbf{e}_1, ...\mathbf{e}_{n-1}$ is modified in such a way that its eigenvalues have different signs. We obtain pseudo-Euclidean geometry. For simplicity we set the dimension $n = 4$ and eigenvalues $g_0 = 1$, $g_1 = g_2 = g_3 = -1$. It is the well known space of Minkowski (pseudo-Euclidean space of index 1). In the space of Minkowski the definition (3.1) of two vectors equality in $\sigma$-representation does not coincide, in general, with the two vectors equality (3.3) in V-representation. The two definitions are equivalent for timelike vectors ($g^{kl}x_k x_l > 0$), and they are not equivalent for spacelike vectors ($g^{kl}x_k x_l < 0$).

Indeed, using definition (3.1), we obtain the relation (3.7) for any vectors of the same length. It means that the vector with coordinates $x_k - y_k$ is an isotropic vector. However the sum, as well as the difference of two timelike vectors of the same length in the Minkowski space is either a timelike vector, or a zeroth vector. Hence, the relations (3.8) take place. Thus, definitions of equality (3.6) and (3.3) coincide for timelike vectors.

In the case of spacelike vectors $x_k$ and $y_k$ their difference $x_k - y_k$ may be an isotropic vector. The relation (3.7) states this. It means that equality of two spacelike vectors of the same length in the $\sigma$-representation of the Minkowski space is multivariant, i.e. there are many spacelike vectors $\mathbf{Q}_0\mathbf{Q}_1, \mathbf{Q}_0\mathbf{Q}_1', ...$ which are equivalent to the spacelike vector $\mathbf{P}_0\mathbf{P}_1$, but the vectors $\mathbf{Q}_0\mathbf{Q}_1, \mathbf{Q}_0\mathbf{Q}_1', ...$ are not equivalent between themselves, in general.

The result is rather unexpected. Firstly, the definition of equality appears to be different in V-representation and in $\sigma$-representation. Secondly, the equality definition in $\sigma$-representation appears to be multivariant, what is very unusual.

In the V-representation the equality of two vectors is single-variant by definition of a vector as an element of the linear vector space. In the $\sigma$-representation the equality of two vectors is defined by the world-function. The pseudo-Euclidean space is a result of a deformation of the proper Euclidean space, when the Euclidean world function is replaced by the pseudo-Euclidean world function. The definition of the vector equality via the world function remains to be the equality definition in all Euclidean spaces.

For brevity we shall use different names for geometries with different definition of the vectors equality. Let geometry with the vectors equality definition (3.3), (3.2) be the Minkowskian geometry, whereas the geometry with the vectors equality definition (3.1) will be referred to as the $\sigma$-Minkowskian geometry. The world function is the same in both geometries. The geometries differ in the structure of the linear vector space and, in particular, in definition of the vector equality. Strictly, there is no linear vector space in the $\sigma$-Minkowskian geometry. The linear vector space is not necessary for formulation of the Euclidean geometry, as well as for formulation of the $\sigma$-Minkowskian geometry. The Minkowskian geometry and the $\sigma$-Minkowskian one are constructed in different representations. The geometry, constructed on the basis of the world function is a more general geometry, because the world function exists in both geometries.

*Important remark.* In application of the space-time geometry of Minkowski to the space-time the spacelike world lines and spacelike vectors are not used. The difference between the geometry of Minkowski and $\sigma$-Minkowskian geometry appears to be unessential from this viewpoint. However, if one considers tachyons, i.e. particles whose world lines are spacelike, one obtains another result [4]

What of the two equality definitions (2.20), or (3.3) are valid? Maybe, both? Apparently, both geometries are possible as abstract constructions. The Minkowskian geometry and the $\sigma$-Minkowskian geometry differ in such a property as the multivariance of spacelike vectors equality. In the geometry, constructed on the basis of the linear vector space, the multivariance is absent in principle (by definition). On the contrary, in the $\sigma$-representation the equivalent vector is determined as a solution of the equations (2.20). For arbitrary world function one can guarantee neither existence, nor uniqueness of the solution. They may be guaranteed, only if the world function satisfies the conditions I $\div$ IV. It means, that the multivariance is a general property of the generalized geometry, whereas the single-variance of the proper Euclidean geometry is a special property of the proper Euclidean geometry. The special properties of the proper Euclidean geometry are described by the conditions I $\div$IV. All these conditions contain a reference to the dimension of the Euclidean space. The single-variance of the proper Euclidean geometry is a special property, which is determined by the form of the world function, but it does not contain a reference to the dimension, because it is valid for proper Euclidean space of any dimension.

From formal viewpoint the $\sigma$-Minkowskian geometry is more consistent, because the definition (3.1) does not depend on the choice of the coordinate system. In the Minkowskian geometry the two vectors equality definition contains a reference to the coordinate system, which may be considered as an additional structure introduced to the geometry of Minkowski. After this the geometry of Minkowski should be qualified as a fortified geometry. It means that a physical geometry is equipped by some additional geometric structure. This structure suppresses multivariance of the spacelike vectors equality.

We may dislike this fact, because we habituated ourselves to single-variance of the two vectors equivalence. However, we may not ignore the fact, that the multivariance is a natural property of geometry. It means that we are to use the two vector equivalence in the form (2.20).

Let us discuss corollaries of the new definition of the vector equivalence for the real space-time. In the Euclidean space the straight line, passing through the points $P_0, P_1$ is defined by the relation

$$\mathcal{T}_{P_0P_1} = \{R|\mathbf{P}_0\mathbf{R} \parallel \mathbf{P}_0\mathbf{P}_1\} \tag{4.1}$$

In the $\sigma$-representation the parallelism $\mathbf{P}_0\mathbf{R} \parallel \mathbf{P}_0\mathbf{P}_1$ of vectors $\mathbf{P}_0\mathbf{R}$ and $\mathbf{P}_0\mathbf{P}_1$ is defined by the relation (2.19) in the $\sigma$-Minkowskian space-time. In the Minkowskian space-time the vector parallelism $\mathbf{P}_0\mathbf{R} \parallel \mathbf{P}_0\mathbf{P}_1$ is defined by four equations, describing proportionality of components of vectors $\mathbf{P}_0\mathbf{R}$ and $\mathbf{P}_0\mathbf{P}_1$. In the $\sigma$-Minkowskian space-time the straight (4.1) is a one-dimensional line for the timelike vector $\mathbf{P}_0\mathbf{P}_1$, whereas the straight (4.1) is a three-dimensional surface (two planes) for the spacelike vector $\mathbf{P}_0\mathbf{P}_1$. In the Minkowskian space-time all straights (timelike and spacelike) are one-dimensional lines. Thus, the straights, generated by the spacelike vector, are described differently in the Minkowskian geometry and in the $\sigma$-Minkowskian one.

The timelike straight in the geometry of Minkowski describes the world line of a free particle, whereas the spacelike straight is believed to describe a hypothetical particle tachyon. The tachyon is not yet discovered. In the $\sigma$-Minkowskian space-time geometry this fact is explained as follows. The tachyon, if it exists, is described by two isotropic three-dimensional planes, which is an envelope to a set of light cones, having its vertices on some one-dimensional straight line. Such a tachyon has not been discovered, because one looks for it in the form of the one-dimensional straight line. In the conventional Minkowskian space-time the tachyon has not been discovered, because there exist no particles, moving with the speed more than the speed of the light. Thus, the absence of tachyon is explained on the geometrical level in the $\sigma$-Minkowskian space-time geometry, whereas the absence of tachyon is explained on the dynamic level in the conventional Minkowskian space-time geometry.

Let us consider the non-Riemannian space-time geometry $\mathcal{G}_{\rm d}$, described by the world function $\sigma_{\rm d}$

$$\sigma_{\rm d} = \sigma_{\rm M} + \operatorname{sgn}(\sigma_{\rm M})\, d, \qquad d = \lambda_0^2 = \frac{\hbar}{2bc} = \text{const} \tag{4.2}$$

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } \ x > 0 \\ 0, & \text{if } \ x = 0 \\ -1, & \text{if } \ x < 0 \end{cases}, \tag{4.3}$$

where $\sigma_{\rm M}$ is the world function of the geometry of Minkowski, $\hbar$ is the quantum constant, $c$ is the speed of the light and $b$ is some universal constant $[b]$ =g/cm. The constant $\lambda_0$ is some universal length.

The space-time geometry (4.2) is discrete , because in this space-time geometry there are no vectors $\mathbf{P}_0\mathbf{P}_1$, whose length $|\mathbf{P}_0\mathbf{P}_1|$ be small enough, i.e.

$$|\mathbf{P}_0\mathbf{P}_1|^4 \notin \left(0, 4\lambda_0^4\right), \qquad \forall P_0, P_1 \in \Omega \tag{4.4}$$

In other words, the space-time geometry (4.2) has no close points. The space-time geometry, where there are no close points should be qualified as a discrete geometry. It is rather unexpected that a discrete space-time geometry may be uniform and isotropic as the geometry (4.2). It is unexpected, that a discrete geometry may be given on a continuous manifold. But this puzzle is connected with the fact, that usually one uses V-representation, where a discrete geometry is given on a lattice space. Another unexpected properties of a discrete space-time geometry can be found in [7].

In the space-time geometry (4.2) a free pointlike particle motion is described by a chain $\mathcal{T}_{\rm br}$ of connected segments $\mathcal{T}_{[P_kP_{k+1}]}$ of straight

$$\mathcal{T}_{\rm br} = \bigcup_k \mathcal{T}_{[P_kP_{k+1}]}, \tag{4.5}$$

$$\mathcal{T}_{[P_kP_{k+1}]} = \left\{ R | \sqrt{2\sigma_{\rm d}(P_k, R)} + \sqrt{2\sigma_{\rm d}(P_{k+1}, R)} = \sqrt{2\sigma_{\rm d}(P_k, P_{k+1})} \right\} \tag{4.6}$$

The particle 4-momentum $\mathbf{p}$ is described by the vector $\mathbf{P}_k\mathbf{P}_{k+1}$

$$\mathbf{p} = bc\mathbf{P}_k\mathbf{P}_{k+1}, \qquad m = b\,|\mathbf{P}_k\mathbf{P}_{k+1}| = b\mu, \qquad k = 0, \pm 1, ... \tag{4.7}$$

Here $m$ is the particle mass, $\mu$ is the geometrical particle mass, i.e. the particle mass expressed in units of length. Description of $\mathcal{G}_\mathrm{d}$ is produced in the $\sigma$-representation, because description of non-Riemannian geometry in the V-representation is impossible. In the geometry $\mathcal{G}_\mathrm{d}$ segments $\mathcal{T}_{[P_k P_{k+1}]}$ of the timelike straight are multivariant in the sense, that $\mathcal{T}_{[P_k P_{k+1}]}$ is a cigar-shaped three-dimensional surface, but not a one-dimensional segment.

For the free particle the adjacent links (4-momenta) $\mathcal{T}_{[P_k P_{k+1}]}$ and $\mathcal{T}_{[P_{k+1} P_{k+2}]}$ are equivalent in the sense of (2.20)

$$\begin{array}{rcl}\mathbf{P}_k\mathbf{P}_{k+1}\mathrm{eqv}\mathbf{P}_{k+1}\mathbf{P}_{k+2} & : & \\ (\mathbf{P}_k\mathbf{P}_{k+1}.\mathbf{P}_{k+1}\mathbf{P}_{k+2}) & = & |\mathbf{P}_k\mathbf{P}_{k+1}| \cdot |\mathbf{P}_{k+1}\mathbf{P}_{k+2}|\,, \qquad |\mathbf{P}_k\mathbf{P}_{k+1}| = |\mathbf{P}_{k+1}\mathbf{P}_{k+2}|\end{array} \tag{4.8}$$

This equivalence is multivariant in the sense that at fixed link $\mathbf{P}_k\mathbf{P}_{k+1}$ the adjacent link $\mathbf{P}_{k+1}\mathbf{P}_{k+2}$ wobbles with the characteristic angle $\theta = \sqrt{d/\mu^2} = \sqrt{b\hbar^2/2m^2c}$. The shape of the chain with wobbling links appears to be random. Statistical description of the random world chain appears to be equivalent to the quantum description in terms of the Schrödinger equation [5]. Thus, quantum effects are a corollary of the multivariance. Here the multivariance is taken into account on the level of the space-time geometry. In conventional quantum theory the space-time geometry is single-variant, whereas the dynamics is multivariant, because, when one replaces the conventional dynamic variable by a matrix or by an operator, one introduces the multivariance in dynamics.

Thus, the multivariance is a natural property of the real world (especially of microcosm). At the conventional approach one avoids the multivariance "by hand" from geometry and introduces it "by hand" in dynamics, to explain quantum effects. It would be more reasonable to remain the multivariance in the space-time geometry, because it appears there naturally. Besides, multivariance of the space-time geometry has another corollaries, other than quantum effects. In particular, multivariance of the space-time geometry is a reason of the restricted divisibility of real bodies into parts (atomism) [6].

# 5 Multivariance and possibility of the geometry axiomatization

In the E-representation of the proper Euclidean geometry one supposes that any geometrical object can be constructed of basic elements (points, segments, angles). In the V-representation any geometrical object is supposed to be constructed of basic elements (points, vectors). Deduction of propositions of the proper Euclidean geometry from the system of axioms reproduces the process of the geometrical object construction. There are different ways of the basic elements application for a construction of some geometrical object, because such basic elements as segments and vectors are simply sets of points. In the analogical way a proposition of the proper Euclidean geometry may be obtained by different proofs, based on the system of axioms.

Let us consider a simple example of the three-dimensional proper Euclidean space. In the Cartesian coordinates $\mathbf{x} = \{x, y, z\}$ the world function has the form

$$\sigma_{\mathrm{E}}\left(\mathbf{x}, \mathbf{x}'\right) = \frac{1}{2}\left(\left(x - x'\right)^2 + \left(y - y'\right)^2 + \left(z - z'\right)^2\right) \tag{5.1}$$

We consider a ball $\mathcal{B}$ with the boundary

$$x^2 + y^2 + z^2 = R^2 \tag{5.2}$$

where $R$ is the radius of the ball. The ball $\mathcal{B}$ may by considered as constructed only of the set $\mathcal{S}$ of the straight segments $\mathcal{T}\left(-x, y, z; x, y, z\right)$, $x^2 + y^2 + z^2 = R^2$

$$\mathcal{B} = \bigcup_{x^2+y^2+z^2=R^2} \mathcal{T}\left(-x, y, z; x, y, z\right) \tag{5.3}$$

Any segment $\mathcal{T}\left(-x, y, z; x, y, z\right)$ is a segment of the length $\sqrt{R^2 - y^2 - z^2}$. Its center is placed at the point $\{0, y, z\}$. The segment $\mathcal{T}\left(x, y, z; -x, y, z\right)$ is the set of points $P'$ with coordinates $\left(x', y, z\right)$

$$\mathcal{T}\left(-x, y, z; x, y, z\right) = \left\{\left\{x' | x'^2 \leq x^2\right\}, y, z\right\} \tag{5.4}$$

Different segments $\mathcal{T}\left(-x, y, z; x, y, z\right)$ have no common points, and any point $P$ of the ball $\mathcal{B}$ belongs to one and only one of segments of the set $\mathcal{S}$.

Let us consider the ball $\mathcal{B}$ with the boundary

$$x^2 + y^2 + z^2 = R^2 - d/2 \tag{5.5}$$

But now we consider non-Riemannian geometry $\mathcal{G}_{\mathrm{d}}$, described by the world function $\sigma_{\mathrm{d}}$

$$\sigma_{\mathrm{d}} = \left\{ \begin{array}{ll} \sigma_{\mathrm{E}} - d, & \text{if} \quad \sigma_{\mathrm{E}} \geq 2d \\ \frac{1}{2}\sigma_{\mathrm{E}}, & \text{if} \quad \sigma_{\mathrm{E}} < 2d \end{array} \right., \qquad d = \text{const}, \; d > 0 \tag{5.6}$$

where $\sigma_{\mathrm{E}}$ is the proper Euclidean world function (5.1). Thus, if $\sigma_{\mathrm{d}} \gg d$ the world function $\sigma_{\mathrm{d}}$ distinguishes slightly from $\sigma_{\mathrm{E}}$.

The straight segment $\mathcal{T}_{[P_0P_1]}$ between the points $P_0$ and $P_1$ is the set of points

$$\mathcal{T}_{[P_0P_1]} = \left\{R | \sqrt{2\sigma_{\mathrm{d}}\left(P_0, R\right)} + \sqrt{2\sigma_{\mathrm{d}}\left(P_1, R\right)} = \sqrt{2\sigma_{\mathrm{d}}\left(P_0, P_1\right)}\right\} \tag{5.7}$$

The segment $\mathcal{T}_{[P_0P_1]}$ is cigar-shaped two-dimensional surface. Let the length of the segment be $l \gg d$ and $\tau$, $0 < \tau < l$ be a parameter along the segment. The radius $\rho$ of the hollow segment tube as a function of $\tau$ has the form

$$\rho^2 = \rho^2\left(\tau\right) = \frac{1}{4}\frac{d}{\left(l - d\right)^2}\left(2\tau - d\right)\left(2l - 3d\right)\left(2l - 2\tau - d\right), \qquad 2d < \tau < l - 2d \tag{5.8}$$

If $\tau \gg 2d$, we obtain approximately

$$\rho^2 = \rho^2\left(\tau\right) = \frac{2d}{l}\tau\left(l - \tau\right), \qquad 2d \ll \tau < l - 2d \tag{5.9}$$

The maximal radius of the segment tube $\rho_{\max} = \sqrt{ld/2}$ at $\tau = l/2$.

It is clear that the ball $\mathcal{B}$ cannot be constructed only of such tube segments. These hollow segments could not fill the ball completely. The problem of construction of the ball, consisting of basic elements (points and segments), appears to be a very complicated problem in the modified geometry $\mathcal{G}_{\rm d}$. This problem of the ball construction associates with the impossibility of deducing the geometry $\mathcal{G}_{\rm d}$ from a system of axioms. It seems that the geometry, deduced from the system of axioms, cannot be multivariant, because any proposition, obtained from axioms by means of the formal logic is to be definite. It cannot contain different versions.

On the other hand, the multivariance is an essential property of the real microcosm. It is a reason of quantum effects and atomism. The space-time geometry is a basis of dynamics in microcosm.

In the multivariant geometry one constructs geometrical objects by means of the deformation principle. Geometrical object $\mathcal{O}_{\rm E}$ is constructed in some region $\mathcal{S}_1$ of the proper Euclidean space. In means that all blocks of $\mathcal{O}_{\rm E}$ as well as the geometrical object $\mathcal{O}_{\rm E}$ itself are expressed in terms of the world function $\sigma_{\rm E}$ of the proper Euclidean geometry. Let us imagine that we need to shift $\mathcal{O}_{\rm E}$ in other region $\mathcal{S}_2$ of the space. We may move all blocks of $\mathcal{O}_{\rm E}$ from $\mathcal{S}_1$ to $\mathcal{S}_2$ and construct a geometrical object $\mathcal{O}'_{\rm E}$ in $\mathcal{S}_2$ from blocks, using the same prescription of construction, which has been used at construction of $\mathcal{O}_{\rm E}$. This prescription, written in terms of the world function, has the same form in any geometry, if one uses only points as the basic concept of the geometry. Points considered as blocks are not deformed at motion from the region $\mathcal{S}_1$ to the region $\mathcal{S}_2$, even if the geometry in the region $\mathcal{S}_2$ distinguishes from the geometry in the region $\mathcal{S}_1$ (different world functions in the regions $\mathcal{S}_1$ and $\mathcal{S}_2$).

This consideration explains application of the deformation principle, but it is not its proof. The deformation principle is the principle which admits one to use physical geometries for description of the nonaxiomatizable space-time geometry

# 6 Concluding remarks

Thus, there are three different representation of the proper Euclidean geometry. In the E-representation there are three basic elements: point, segment, angle. The segment and the angle are some auxiliary structures. The segment is determined by two points. The angle is determined by three points, or by two connected segments. In V-representation there are two basic elements: point and vector (directed segment). The angle is replaced by two segments (vectors). Its value is determined by the scalar product of two vectors. Reduction of the number of basic elements is accompanied by appearance of a new structure: linear vector space with the scalar product, given on it. Information, connected with the angle, is concentrated now in the scalar product and in the linear vector space. Such a concept as distance is a property of a vector (or a property of two points).

In the $\sigma$-representation there is only one basic element: point. Interrelation of two points (segment) is described by the world function (distance). Mutual directivity of segments, (or an angle) is considered as an interrelation of three points. It is described by means of the scalar product, expressed via distance (world function). In $\sigma$-representation of Euclidean geometry the world function (distance) turns into a structure in the sense, that the world function satisfies a series of constraints (conditions I $\div$ IV). The concept of distance exists in all representations. But in the E-representation and in the V-representation the distance is not considered as a structure, because the conditions I $\div$ IV are not considered as constraints, imposed on the distance (world function). Of course, these conditions are fulfilled in all representations of Euclidean geometry, but they are considered as direct properties of the proper Euclidean space, but not as constraints, imposed on distance of the proper Euclidean space.

The $\sigma$-representation is interesting in the sense, that it contains only one basic element (point) and only one structure (world function). All other concepts of Euclidean geometry appear to be expressed via world function. Supposing that these expressions have the same form in other geometries, one can easily construct them, replacing world function. This replacement looks as a deformation of the proper Euclidean space.

Usually a change of a representation is a formal operation, which is not accompanied by a change of basic concepts. For instance, representations in different coordinate systems differ only in the form of corresponding algebraic expressions. A change of representation of the proper Euclidean geometry is accompanied by a change of basic concepts, when the primary concepts turns to the secondary ones and vice versa. The procedure of a change of representation may be qualified as the logical reloading. The logical reloading is rather rare logical procedure. Such a change of primary concepts is unusual and difficult for a perception.

In particular, such difficulties of perception appear, because the linear vector space is considered as an attribute of the geometry (but not as an attribute of the geometry description). The linear vector space is an attribute of the V-representation of the proper Euclidean geometry. There are geometries (physical geometries), where the linear vector space cannot be introduced at all. Physical geometries are described completely by the world function, and the world function is the only characteristic of the physical geometry. All other attributes of the physical geometry are derivative. They can be introduced only via the world function. The physical geometry cannot be axiomatized, in general. The proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is an unique known example of the physical geometry, which can be axiomatized. Axiomatization of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is used for construction of $\mathcal{G}_{\mathrm{E}}$. When the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is constructed (deduced from the Euclidean axiomatics), one uses the fact that the $\mathcal{G}_{\mathrm{E}}$ is a physical geometry. One expresses all geometrical objects of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ in terms of the Euclidean world function $\sigma_{\mathrm{E}}$. Replacing $\sigma_{\mathrm{E}}$ in all definitions of the $\mathcal{G}_{\mathrm{E}}$ by another world function $\sigma$, one obtains all definitions of another physical geometry $\mathcal{G}$. It means that one obtains another physical geometry $\mathcal{G}$, which cannot be axiomatized (and deduced from some axiomatics)

Impossibility of the geometry $\mathcal{G}$ axiomatization is conditioned by the fact, that the equivalence relation is intransitive, in general, in the geometry $\mathcal{G}$. However, in any mathematical model, as well as in any geometry, which can be axiomatized, the equivalence relation is to be transitive. Almost all mathematicians believe, that any geometry can be axiomatized. Collision of this belief with the physical geometry application generates misunderstandings and conflicts [7]. Axiomatizability is a property of the Euclidean method of the **geometry construction**, but not a property of the Euclidean **geometry itself**. Using another method of the geometry construction (method of the geometry deformation), one may construct nonaxiomatizable geometries.

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